Brutal Force

Po — MOP '98

This is probably going to be my final lecture at MOP (giving or receiving), so here's what I'm leaving for the rest of you.

I'm showing how to use *brutal force* to solve geometry problems. *Brutal force* is just what I'm calling my method of twisting/smushing because hey, twisting/smuching a diagram is exerting brutal force on it.

Now that my neat little introduction is over, let's get to the point. What exactly is this *brutal force* and how do I use it?

There are many ways to deform a diagram. Here are the basic tools:

Reflection This is one of the most basic operations. It involves finding a line in the plane and reflecting [part] of the problem around it, possibly superimposing the two images. (Duh)

Rotation This is another basic operation. Very useful in circles (for the obvious reasons).

Homothety I hope you know what this is. This is fun because stuff stays similar and that's cool.

Spiral Similarilty This is a useful tool later, when we do much more drastic stuff. But what it really does is rotate something, and then homothety it around somewhere.

All right! Now that you know that, let's find out how to do some more advanced stuff.

Look for spinning stuff In a picture, sometimes there's a hidden spirally-similar figure. Look closely, and you may be able to rotate/homothety/spiral-similar something.

Stretch Here we go. This is more interesting. Stretching stuff is (kinda) like homothety-ing something in one dimension. A nice fact is that stuff always travels in one direction, emabling you to do lots of cool stuff. Important: everything moves by the same ratio.

Twist This involves moving a part of the diagram along an arc of a circle. The neat thing that it does is preserve angles.

Pull This involves moving one part of the diagram along a ray, and then adjusting the rest of the picture to abide by that deformation.

Those usually work. But by far the most fun operations to perform are the things that generalize facts.

"Induction" on geometrical figures This is the coolest (and also the most difficult) algorithm to perform on a picture. What you do is:

1. Find an easy (usually symmetric) case for the problem. For example, if the problem is about a triangle, show that it's true for an isosceles triangle. This is the base case.
2. Find a line/point/circle that the entire problem depends on. Treat the problem as a function: the input is the position of that crucial line/point/circle and the output is the status of the problem (T/F . . . hopefully it's true).
3. Stretch/Twist/Pull that crucial element by an incremental amount (ϵ, θ, d, etc.).
4. Superimpose the original and deformed figures (use at least two colors!).
5. Assuming the base case, show that the problem is still true for this deformed figure.
6. By Po's Geometrical Induction Theorem (also the *If you can't see this you don't deserve to be reading this* Theorem), the problem is universally true.

Now that you've got the tools, let's solve some problems!

1. (MOP Test #3, Problem #3) Let $\triangle ABC$ be inscribed in circle O. Let X be the intersection of BO and circle O. Let Y be the midpoint of AC. Choose an arbitrary D on arc AC. Construct equilateral triangles CDF and ADE outwards. Let M be the midpoint of EF. Prove that $XY = XM$.

2. (Razvan, 6/19, *Quadrilaterals* #6) Prove that if in a convex quadrilateral two opposite angles are congruent, the bisectors of the other two angles are parallel.

3. (Rasvan, 6/19, *Quadrilaterals* #13) Prove that the interior bisectors of the angles of a parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram.

4. (Elgin, 6/30, *Complex Numbers* #4) Let A, B, C be distinct points on a line and α be a given angle with $0 < \alpha < 180$. Isosceles triangles ABX and BCY, with $AX = BX$, $CY = BY$ and angle $AXB = \alpha =$ angle BYC are drawn so that their cyclic orientations A-B-X and B-C-Y are both counterclockwise. Then isosceles triangle XYZ is drawn with $XZ = YZ$, angle $XZY = \alpha$, and clockwise orientation X-Y-Z. Prove that Z is on the line containing A, B, and C.

5. (MOP Test #9, Problem #5) Given a triangle ABC, construct the circle with diameter AB. Let the circle intersect AC and BC in D and E. Let the feet of the perpendiculars from D and E to AB be F and G. Let the intersection of DG and EF be M. Prove that CM is perpendicular to AB.

6. (Rookie Contest, Po's Star Theorem) Given two congruent circles, ω_1 and ω_2. Let them intersect at B and C. Select a point A on ω_1. Let AB and AC intersect ω_2 at A_1 and A_2. Let X be the midpoint of BC. Let A_1X and A_2X intersect ω_1 at P_1 and P_2. Prove that $AP_1 = AP_2$.

7. (Razvan, 6/11, *Triangles* #3) In the triangle ABC we consider the bisector AA'. The circumcircles of the triangles ABA' and ACA' intersect AC and AB in M and N. Prove that $|BN| = |CM|$.

8. (Po, random problem) Let us have three concurrent circles ω_1, ω_2, and ω_3. Let the point of concurrence be P. Let the intersection of ω_1 and ω_2 be A, the intersection of ω_2 and ω_3 be B, and the other intersection be C. Select a point X on ω_1. Let the intersection of XA and ω_2 be Y. Let the intersection of YB and XC be Z. Prove that:

   (a) Z lies on ω_3.

   (b) XY/XZ is a constant.

9. (IMO Test #3, Problem #3) Let circle ω_1, centered at O_1, and circle ω_2, centered at O_2, meet at A and B. Let l be a line through A meeting ω_1 again at Y and meeting ω_2 again at Z. Let X be the intersection of the tangent to ω_1 at Y and the tangent to ω_2 at Z. Let ω be the circumcircle of O_1O_2B, and let Q be the second intersection of ω with BX. Prove that the length of XQ equals the diameter of ω.

10. (APMO '98, Problem #4) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

11. (Lots of other problems) Go look up any geometry problem and try to induct it.