11. Integer polynomials

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CMU Putnam Seminar, Fall 2024

1 Classical results

Vandermonde determinant. Let $a_0, a_1, \ldots, a_n$ be distinct numbers, and let $b_0, b_1, \ldots, b_n$ be arbitrary (possibly equal to each other or to any of the a_i). Then there is a unique polynomial $p(x) = c_n x^n + \cdots + c_0$ such that $p(a_i) = b_i$ for all $0 \le i \le n$.

Lagrange interpolation. An expression for the above polynomial is

$$p(x) = \sum_{i=0}^{n} \frac{b_i}{\prod_{j \neq i}(a_i - a_j)} \prod_{j \neq i}(x - a_j)\,.$$

Fermat's Last Theorem. The equation $x^n + y^n = z^n$ has no positive integer solutions (x, y, z, n) with $n \ge 3$.

Eisenstein's Criterion. Suppose that the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, and there is a prime number p such that p divides all of a_0, a_1, $\ldots$, a_{n-1}, p does not divide a_n, and p^2 does not divide a_0. Then the polynomial is irreducible over the rational numbers.

2 Problems

1. Let $p(x)$ be the polynomial $(x-a)(x-b)(x-c)(x-d)$. Assume $p(x) = 0$ has four distinct integral roots and that $p(x) = 4$ has an integral root k. Show that k is the mean of a, b, c, d.

2. Let $p(x)$ be a polynomial with integer coefficients. Suppose that for some positive integer c, none of $p(1), p(2), \ldots, p(c)$ are divisible by c. Prove that $p(b)$ is not zero for any integer b.

3. Let $p(x)$ be a polynomial with the property that for every $z \in \mathbb{Z}^+$, $p(z)$ is an integer. Prove that for every $z \in \mathbb{Z}$, $p(z)$ is an integer.

4. Let $p(x)$ be a real polynomial of degree n such that $p(m)$ is integral for all integers m. Show that if k is a coefficient of $p(x)$, then $n!k$ is an integer.

5. Suppose that polynomial $P(x)$ has the property that the set

$$\{P(x) : x \in \mathbb{Q}\}$$

includes all rational numbers. Prove that P has degree 1.

6. Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

7. For what positive integers n does the polynomial $p(x) = x^n + (2+x)^n + (2-x)^n$ have a rational root?

8. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and suppose that each a_n is 0 or 1.

(1) Show that if $f(1/2)$ is rational, then $f(x)$ has the form $p(x)/q(x)$ for some integer polynomials $p(x)$ and $q(x)$.

(2) Show that if $f(1/2)$ is not rational, then $f(x)$ does not have the form $p(x)/q(x)$ for any integer polynomials $p(x)$ and $q(x)$.

9. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.