11. Integer Polynomials

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1 Classical results

Well-known fact. Let $P(n)$ be a polynomial with integer coefficients, and let a and b be integers. Show that $P(a) - P(b)$ is divisible by $a - b$.

Gauss. If a polynomial with integer coefficients can be factored into a product of polynomials with rational coefficients, then it can also be factored into a product of polynomials with integer coefficients.

Eisenstein. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial, such that there is a prime p for which

(i) p divides each of a_0, a_1, $\ldots$, a_{n-1},

(ii) p does not divide a_n, and

(iii) p^2 does not divide a_0.

Then $P(x)$ cannot be expressed as the product of two non-constant polynomials with integer coefficients.

Integers. There is a polynomial which takes integer values at all integer points, but does not have integer coefficients.

Rational Root Theorem. Suppose that $P(x) = a_n x^n + \cdots + a_0$ is a polynomial with integer coefficients, and that one of the roots is the rational number p/q (in lowest terms). Then, $p \mid a_0$ and $q \mid a_n$.

2 Problems

1. What is the largest positive integer that is a factor of $P(1) - 2P(7) + P(13)$, for every polynomial P with integer coefficients?

2. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

3. Prove that for every prime number p, the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

cannot be expressed as the product of two non-constant polynomials with integer coefficients.

4. Suppose that the polynomial $P(x)$ with integer coefficients takes values ± 1 at three different integer points. Prove that it has no integer zeros.

5. Let $P(x)$ be a polynomial with integer coefficients. Suppose that there is an integer a for which $P(P(\cdots P(a) \cdots)) = a$, where P is iterated some number of times which is at least twice. Then, $P(P(a)) = a$.

6. Let $P(x)$ be a polynomial with integer coefficients which cannot be factored as a product of polynomials with integer coefficients. Prove that $P(x)$ has no multiple roots.

7. Let $P(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $P(x)$ cannot be expressed as the product of two non-constant polynomials with integer coefficients.

8. Suppose $q_0, q_1, q_2, \dots$ is an infinite sequence of integers satisfying the following two conditions:

   (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,

   (ii) there is a polynomial P and an integer Δ such that $|q_n - P(n)| < \Delta$ for all n.

   Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

9. For every polynomial $P(x)$ with integer coefficients, does there always exist a positive integer k such that $P(x) - k$ is irreducible over integers?

10. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}$$