10. Combinatorics

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1 Classical results

Designs. There are $2n$ students at a school, for some integer $n \geq 2$. Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

Catalan numbers. Find a closed-form expression for the number of valid sequences containing n pairs of parantheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

Partitions. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also, let $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

2 Problems

1. Given two sets A and B, let the notation $A \oplus B$ denote the *symmetric difference* of A and B, i.e., the set of all elements in exactly one of A or B. Express $|A_1 \oplus A_2 \oplus \cdots \oplus A_n|$ in terms of $|A_i|$, $|A_i \cap A_j|$, $|A_i \cap A_j \cap A_k|$, etc., along the lines of the Inclusion-Exclusion formula.

2. Express $|A_1 \cap A_2 \cap \cdots \cap A_n|$ in terms of $|A_i|$, $|A_i \cup A_j|$, $|A_i \cup A_j \cup A_k|$, etc., along the lines of the Inclusion-Exclusion formula.

3. Red and Blue are playing a game on a graph in which all degrees are 100. They take turns, each choosing a single edge to color with their name. Once an edge is chosen by some player, it can never be chosen again. Show that each player has a strategy which ensures that no matter how the other player plays, when all edges are colored, every vertex is incident to at least 25 blue edges.

4. Consider a circular necklace with 2013 beads, each of which is painted either white or green. Call a painting "good" if, among any 21 successive beads, there is at least one green bead. Prove that the number of good paintings of a necklace is odd. **Note:** here, two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.

5. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \ldots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the

selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

6. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.