7. Convergence

Po-Shen Loh

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1 Classical results

Harmonic series. Without using Calculus, show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Infinite product. Suppose (a_n) is a sequence of numbers which satisfies $\sum_{n=1}^{\infty} |a_n| < \infty$. Then the product $(1+a_1)(1+a_2)(1+a_3)\cdots$ tends to a nonzero limit.

Alternating series. Let (a_n) be a monotonic decreasing sequence of positive real numbers which converges to 0. Then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

2 Problems

1. Is the series
$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$$
convergent or divergent?

2. Is the series
$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$
convergent or divergent?

3. Is the series
$$\sum_{n=100}^{\infty} \frac{1}{n(\log n)(\log \log n)}$$
convergent or divergent?

4. Show that there is a rearrangement of the fractions $-\frac{1}{1}$, $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{4}$, etc., into a sequence $a_1, a_2, a_3, \ldots$, such that their limit of partial sums $a_1 + a_2 + \cdots + a_n$ is π.

5. Let (a_n) be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let (b_n) be a rearrangement of the sequence such that for every non-negative integer m, the terms b_{3m+1}, b_{3m+2}, b_{3m+3} are a rearrangement of the terms a_{3m+1}, a_{3m+2}, a_{3m+3} (thus, for example, the first 6 terms of the sequence (b_n) could be $a_3, a_2, a_1, a_4, a_6, a_5$). Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

6. Determine whether the series
$$\sum_{n=2}^{\infty} \frac{1}{n^{1+(\log \log n)^{-2}}}$$
is convergent or divergent. (The logarithm is in base e.)

7. Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left|1 - \frac{a_{n+1}}{a_n}\right|$ is divergent.

8. Find
$$\lim_{x\to\infty} (2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x.$$

9. Let $(a_n)_{n>1}$ be an infinite sequence with $a_n > 0$ for all n. For $n > 1$, let b_n denote the geometric mean of $a_1, \ldots, a_n$, that is, $\sqrt[n]{a_1 \cdots a_n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

10. Define a sequence by $a_1 = 1$, $a_2 = \frac{1}{2}$, and $a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2}$ for every positive integer n. Find
$$\lim_{n\to\infty} n a_n.$$

11. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i), and define $b_n = \frac{1}{n a_n^2}$. Prove that
$$\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n}$$
is convergent.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.