6. Inequalities

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1 Classical results

AM-GM. For any non-negative reals $x_1, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n}.$$

Rearrangement. For any reals $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, and any reordering $y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}$,

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Cauchy-Schwarz. For any reals $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \le (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Jensen. For any convex function f, and any reals $x_1, \ldots, x_n$,

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

2 Problems

1. Several gas stations are located along a circular road. Among them, there is just enough gas for one car to complete a single trip around the circle. Is it always true that there is always a place where you can start, so that your car can make it all the way around once?

2. A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

3. The sum of the squares of two real numbers is 4. What is the smallest possible value of the sum of their fourth powers?

4. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

5. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$\frac{a_1 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

6. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \le k \le n$ is *good* if there exists some ℓ with $1 \le \ell \le m$ such that

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \ge 0,$$

where the indices are taken modulo n. Let T be the set of all good indices. Prove that

$$\sum_{k \in T} a_k \ge 0.$$

7. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimize the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.