21-228 Discrete Mathematics

Assignment 7

Due Fri Apr 28, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Is the following statement true for every positive integer n?

   Every graph consisting of two edge-disjoint Hamiltonian paths contains a Hamiltonian cycle.

2. Let $R(t, t, t)$ be the smallest integer n such that every 3-coloring of the edges of K_n contains a monochromatic K_t. Prove that

   (a) $R(t, t, t) \leq 27^t$, and

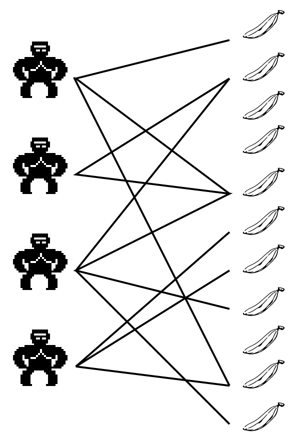
   (b) derive an exponential lower bound for the 3-color Ramsey number $R(t, t, t)$. That is, for each t, come up with an n_t, and a 3-coloring of the edges of K_{n_t} such that there is no monochromatic K_t, where n_t grows exponentially with t.

   Note: I can do this with $n_t = 3^{t/2}$ for every even t (which is sufficient), but it is OK if you prove this with a different n_t, as long as yours grows exponentially with t.

3. There are 100 cities in a country, called "1", "2", . . . , "100". The government would like to connect them all with a network, and it costs $\max\{|i - j|, 4\}$ to lay a wire between cities "i" and "j". (There is a minimum cost of 4 per wire.) Another company comes along, offering to create microwave links between city pairs at a cost of 2 each, with the catch that their technology can only connect city pairs of the form ("i", "$2i$"). For example, it would cost 30 to lay a wire between cities 31 and 61, 4 to lay a wire between cities 31 and 32, and 2 to microwave-link 30 and 60. What is the minimum cost to create this connected network?

4. Let G be an arbitrary graph. It will always have spanning subgraphs (containing all of the vertices) which are bipartite. Among all of them, pick one which has the maximum number of edges, and call this subgraph H. This is called a maximum bipartite subgraph of G. The bipartition divides the vertices of H (and G) into two sides, L and R. Prove that every vertex v has the property that (according to G) at least half of its incident edges go across to the other side. For example, if v were in L, and v happened to have degree 10 in G, then at least 5 of those edges would have their other endpoint in R.

5. King Kong and his friends are sharing some bananas. Each gorilla has a particular subset of the bananas that he/she is interested in. (Perhaps some of the others are mushy.) Coincidentally, it turns out that for each set of S gorillas, the collection of bananas that any of them are interested in is of size at least $2|S|$. Prove that there is a way to give two bananas to each gorilla, so that every gorilla receives only bananas that he/she was interested in.



In graph theoretic language: Let G be a bipartite graph with bipartition $V = A \cup B$. Suppose that for every set $S \subset A$, its combined neighborhood satisfies $|N(S)| \geq 2|S|$. Here, $N(S)$ the collection of all $w \in B$ for which w has a neighbor in S. It is not necessary for every vertex in S to be adjacent to w. Then, prove that G admits a perfect 1-to-2 matching, i.e, a choice of two neighbors for every $v \in A$, with no vertex in B being assigned twice.