21-228 Discrete Mathematics

Assignment 6

Due Fri Apr 14, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Let G be a graph in which all vertices have degree at least 10. Prove that G contains a cycle with at least 11 vertices.

2. Let G be a connected graph with 100 vertices, in which all vertices have degree at least 10. Prove that G contains a path with at least 21 vertices.

3. A country has 100 cities, and some pairs of the cities are linked by highways. Every city is reachable from every other city through some sequence of roads. The government would like to make some of the inter-city links into toll roads. Prove that it is possible to do this in such a way that every city has an odd number of toll roads incident to itself. Assume that highways merge only at cities.

   In graph-theoretical language: prove that given any connected 100-vertex graph, it is always possible to select some of the edges such that every vertex is incident to an odd number of selected edges.

4. A graph is called *triangle-free* if it does not have any three distinct vertices u, v, w where uv, vw, wu are all edges. An *independent set* of size t is a collection of vertices $v_1, \ldots, v_t$ such that none of $v_i v_j$ are edges. Prove that there are numbers c and n_0 such that the following holds: every n-vertex triangle-free graph with $n \geq n_0$ has an independent set of size $c\sqrt{n}$. (Much more is true.)

   Hint: $c = 1.414$ will work, but you may use a different c if you wish.

5. A *dominating set* in a graph is a subset S of vertices such that every vertex not in S is adjacent to at least one vertex in S. The *codegree* of a pair of vertices x, y is the number of vertices z for which both xz and yz are edges.

   Let G be a graph with $3n$ vertices, with the property that every pair of vertices has codegree at least 1. That is, $\forall x \forall y \exists z$ such that xz and yz are both edges. Show that G has a dominating set of size n.

   BTW, it's actually possible to find a much smaller dominating set, using a more advanced technique.