

## 21-228 Discrete Mathematics

### Assignment 4

Due Mon Mar 3, at start of class

**Notes:** Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Consider the first  $3^n$  rows of Pascal's triangle. These are the rows from  $\binom{0}{0}$ , which has single number in a row, to  $\binom{3^n-1}{0}, \dots, \binom{3^n-1}{3^n-1}$ , which has  $3^n$  numbers in a row. For each row, count the number of entries which are **not** divisible by 3. If the number of such entries in a row is a prime power (expressible as  $p^k$  for some prime  $p$  and some positive integer  $k \geq 1$ ), then we say that the row is *cool*. How many of the first  $3^n$  rows are cool? Find a general formula in terms of  $n$ . It's a nice formula, expressible using only arithmetic and power operations, and without any ellipses or summation ( $\sum$ ) or product ( $\prod$ ) notation.

For example, if  $n = 1$ , then we are looking at the first three rows. The first row is just "1", and so it has 1 entry which is not divisible by 3. As 1 is not a prime power, this row is not cool. The second row is "1 1", with 2 entries that are not divisible by 3. Since 2 is a prime power, this row is cool. The third row is "1 2 1", with 3 entries (a prime power) non-divisible by 3, hence it is also cool. Therefore, if  $n = 1$ , then 2 out of the first  $3^1$  rows are cool. As a check for your formula, it is known that if  $n = 4$ , then 30 out of the first  $3^4$  rows are cool.

2. The Jacobsthal numbers are defined by the recursion  $a_n = a_{n-1} + 2a_{n-2}$  with initial conditions  $a_1 = 1, a_2 = 3$ . Prove that

$$a_n = \text{round} \left\{ \frac{2^{n+1}}{3} \right\},$$

for every nonnegative integer  $n$ . Here,  $\text{round}(x)$  denotes the nearest integer to  $x$ , rounding up if  $x$  is a half-integer. For example,  $\text{round}(1.1) = 1 = \text{round}(0.9)$  and  $\text{round}(1.5) = 2$ .

3. Find an explicit formula for the recursion defined by  $a_n = 2a_{n-1} - 2a_{n-2}$  with initial conditions  $a_0 = 0$  and  $a_1 = 1$ .
4. In class, we handled the case when matrices were diagonalizable. This exercise guides you through the case when the matrix is not!

Find an explicit formula for the solution to the recurrence  $a_n = 4a_{n-1} - 4a_{n-2}$ , with initial conditions  $a_0 = 0$  and  $a_1 = 1$ . Please use the following (rather than guessing the formula and using induction):

- (Jordan Canonical Form.) There exists a  $2 \times 2$  matrix  $P$  for which

$$\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} = P \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} P^{-1}.$$

- (Power of elementary Jordan block.) For any  $\lambda$  and any positive integer  $n$ :

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}.$$

5. Consider the generating function

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that for each integer  $n \geq 0$ ,

$$a_n^2 + a_{n+1}^2 = a_{2n+2}.$$

**Hint:** Find a  $2 \times 2$  matrix  $A$  such that

$$A^{n+2} = \begin{bmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{bmatrix},$$

and consider the top left entry of the matrix product  $A^{n+2}A^{n+2}$ .