Putnam Σ.10

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1 Problems

Putnam 2011/A4. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Putnam 2011/A5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbb{R}^2$, the vector $\nabla F(u, v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v) \rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \ldots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

Putnam 2011/A6. Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, ..., g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m \to \infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2$$

is positive and finite.