Putnam Σ.8

Po-Shen Loh

23 October 2022

1 Problems

Putnam 2009/B4. Say that a polynomial with real coefficients in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

Putnam 2009/B5. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

Putnam 2009/B6. Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \leq (b \bmod c) < c$.]