Putnam Σ.6

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1 Problems

Putnam 2007/B4. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

Putnam 2007/B5. Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\leq a$.)

Putnam 2007/B6. For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \leq k \leq n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \leq f(n) \leq n^{n^2/2 + Cn} e^{-n^2/4}.$$