13. Geometry

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1 Problems and well-known statements

1. Let P be a point inside a continuous closed curve in the plane which does not intersect itself. Show that there are two points on the curve whose midpoint is P.
2. Let convex quadrilateral $ABCD$ be given in a plane, and let X be a point not on the plane. Show that there are points A', B', C', and D' on the lines XA, XB, XC, and XD, respectively, with the property that $A'B'C'D'$ is a parallelogram.
3. Given any bounded plane region, prove that there are three concurrent lines that cut it into six pieces of equal area.
4. Given any bounded plane region, prove that there is a point through which no line trisects the area.
5. Given a finite collection of closed squares of total area 3, prove that they can be arranged to cover the unit square.
6. Given a finite collection of closed squares of total area $\frac{1}{2}$, prove that they can be arranged to fit in the unit square (with no overlaps).
7. Let OA and OB be two rays in the plane, and let P be a point between them. Which point X on the ray OA has the property that if XP is extended to meet the ray OB at Y, then $XP \cdot PY$ is minimized?
8. Given a region whose boundary is a simple polygon of area a and perimeter p, prove that it contains a disc with radius larger than a/p.
9. Given a right triangle and a finite set of points inside it, prove that these points can be connected by a path of line segments, such that the sum of squares of segment lengths in this path is at most the square of the hypotenuse.
10. Let an ellipse have center O and foci A and B. For a point P on the ellipse, let d be the distance from O to the line of tangency to the ellipse at P. Show that $PA \cdot PB \cdot d^2$ is independent of the position of P.

2 Homework

There is no homework due next week. Don't forget to come to the Putnam on Saturday!