11. Integer Polynomials

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CMU Putnam Seminar, Fall 2022

1 Problems and well-known statements

1. (Goldbach.) Prove that there is no polynomial $P(x)$ with integer coefficients and degree at least 1, such that $P(0)$, $P(1)$, $P(2)$, $\ldots$ are all prime.

2. If P is a polynomial with integer coefficients, and a and b are distinct integers, then $P(a) - P(b)$ is divisible by $a - b$.

3. Let $P(x)$ be a polynomial such that $P(n)$ is an integer for every integer n. (Note that the coefficients of P are not necessarily integers themselves.) Prove that there are some integers $c_0, \ldots, c_n$ for which

$$P(x) = c_0 \binom{x}{0} + c_1 \binom{x}{1} + \cdots + c_n \binom{x}{n},$$

   where $\binom{x}{k}$ is defined for all real x to be $\frac{1}{k!} x(x-1)(x-2)\cdots(x-k+1)$.

4. I'm thinking of a polynomial P with nonnegative integer coefficients. As many times as you wish, you're allowed to give me a real number a, and I will evaluate $P(a)$ and tell you the result. Can you figure out what P is (as a polynomial), and if so, how few guesses can you achieve this in?

5. Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that the conditions $P(a) = b$, $P(b) = c$, and $P(c) = a$ cannot be satisfied simultaneously.

6. Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P(P(\cdots P(x)\cdots)) = x$ for some integer x, where P is repeated n times, then $P(P(x)) = x$.

7. Let $P(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$, where a, b, c, d, e are integers and $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is also rational.

8. What is the lowest degree monic polynomial (i.e., with leading coefficient equal to 1) for which $P(n) \equiv 0 \pmod{100}$ for every integer n?

9. Let $p(x) = x^3 + ax^2 + bx - 1$ and $q(x) = x^3 + cx^2 + dx + 1$ be polynomials with integer coefficients. Suppose that $p(x)$ is irreducible over the rationals, and α is a root of $p(x) = 0$, and $\alpha + 1$ is a root of $q(x) = 0$. Find an expression for another root of $p(x) = 0$ in terms of α, but not involving a, b, c, or d.

10. Let α be a complex $(2^n + 1)$-th root of unity. Prove that there always exist polynomials $p(x)$ and $q(x)$ with integer coefficients, such that

$$p(\alpha)^2 + q(\alpha)^2 = -1.$$

11. Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

    is an integer, and determine its value.

2 Homework

Please write up solutions to two of the statements/problems, to turn in at next week's meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.