

# 10. Combinatorics

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## 1 Well-known statements

**Catalan number / Reflection principle.** The  $n$ -th Catalan number  $C_n$  counts the number of ways to arrange  $n$  pairs of parantheses in a valid expression. For example,  $((())())$  is a valid expression, but  $(())(())$  is not. This number turns out to be exactly

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

**Binomial theorem.** For any integer  $n$ ,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

## 2 Problems

1. Suppose there are many people at a party, but each person knows at most 7 other people at the party. Show that it's possible to separate the people into two rooms such that in each room, every person knows at most 3 of the people in their room.
2. The points of the plane are each colored either red, yellow, or blue. Prove that there are two points of the same color having mutual distance 1.
3. The points of the plane are each colored either red or blue. Prove that one of these colors contains pairs of points at *every* mutual distance.
4. Good coins weigh 10 grams, and bad ones weigh 9 grams. Given 4 coins and a scale (not a balance, but a scale that reports the total weight of a pile of coins), determine what type each coin is in only 3 weighings.
5. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can be participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.
  - (i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.
  - (ii) At any moment, he may double the whole family of imons in his lab by creating a copy  $I'$  of each imon  $I$ . During this procedure, the copies  $I'$  and  $J'$  become entangled if and only if the original imons  $I$  and  $J$  are entangled, and each copy  $I'$  becomes entangled with its original imon  $I$ ; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

6. Compute

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (x-k)^n.$$

7. Compute the inverse of the following matrix:

$$\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n}{0} \\ \binom{0}{1} & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{0}{n} & \binom{1}{n} & \binom{2}{n} & \cdots & \binom{n}{n} \end{pmatrix},$$

where  $\binom{n}{k} = 0$  when  $n < k$ .

8. I choose an integer from 0 through 15. You ask me 7 yes or no questions. I answer them all, but I am allowed to *lie* once. (I don't have to, but I am allowed to.) Determine my number!
9. Suppose that each vertex of a graph is equipped with an indicator light and a button. Each vertex's button simultaneously toggles the states of all of its neighbors, as well as its own state. Initially, all lights are off. Prove that it is possible to turn on all of the lights.
10. The vertex set of any graph can be partitioned into two (possibly empty) sets such that each set induces a subgraph with all degrees even.

### 3 Homework

Please write up solutions to two of the statements/problems, to turn in at next week's meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.