9. Linear Algebra

Po-Shen Loh

CMU Putnam Seminar, Fall 2022

1 Well-known statements

Integer matrices. A square matrix with all-integer entries has inverse consisting of all-integer entries if and only if its determinant is ± 1.

Area. If a triangle in the plane has coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), then its area is the absolute value of:

$$\frac{1}{2} \cdot \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

Spectral mapping theorem. If an $n \times n$ square matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly with multiplicity), and $P(x)$ is a polynomial, then the eigenvalues of the matrix $P(A)$ are $P(\lambda_1), \ldots, P(\lambda_n)$.

Commuting, sort of. For an $n \times n$ matrix A, let $\phi_k(A)$ denote the degree-k symmetric polynomial in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A:

$$\phi_k(A) = \sum_{i_1, i_2, \ldots, i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

For example, $\phi_1(A)$ is the trace of A, and $\phi_n(A)$ is the determinant of A. Prove that for every $1 \le k \le n$, and every pair of $n \times n$ matrices A and B,

$$\phi_k(AB) = \phi_k(BA).$$

2 Problems

1. For any $n \times n$ matrix A with real entries,

$$\det(I_n + A^2) \ge 0.$$

2. Let A, B, and D be $n \times n$ matrices. Prove that

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(AD).$$

3. Let A, B, C, and D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

4. Let X and Y be $n \times n$ matrices, and let I_n be the $n \times n$ identity matrix. Prove that

$$\det(I_n - XY) = \det(I_n - YX).$$

5. There are given $2n+1$ real numbers, $n \geq 1$, with the property that whenever one of them is removed, the remaining $2n$ can be split into two sets of n elements that have the same sum of elements. Prove that all the numbers are equal.

6. Let A be an $n \times n$ matrices such that a_{ij} is the entry in the i-th row and j-th column. Suppose that for every row i, $\sum_{j=1}^{n} |a_{ij}| < 1$. Prove that $I_n - A$ is invertible.

7. Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.

8. Let $k < n$ be two positive integers. Compute:

$$\det \begin{pmatrix} \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{0} & \binom{n+k}{1} & \cdots & \binom{n+k}{k} \end{pmatrix}$$

9. Given distinct integers $x_1, x_2, \ldots x_n$, prove that $\prod_{i<j}(x_i - x_j)$ is divisible by $1!2!\cdots(n-1)!$.

3 Homework

Please write up solutions to two of the statements/problems, to turn in at next week's meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.