7. Convergence

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1 Well-known statements

Infimum and supremum. The *infimum* of a set of numbers $S \subseteq \mathbb{R}$ is the largest real number x such that $x \le s$ for all $s \in S$. The *supremum* is the smallest real number y such that $y \ge s$ for all $s \in S$.

Monotone sequence. Let $a_1 \ge a_2 \ge a_3 \ge \cdots$ be a sequence of non-negative numbers. Then $\lim_{n\to\infty} a_n$ exists.

Alternating series. Let $a_1, a_2, \ldots$ be a sequence of real numbers which is decreasing in absolute value, converging to 0, and alternating in sign. Then the series $\sum_{i=1}^{\infty} a_i$ converges.

Conditionally convergent. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that the series $\sum_{i=1}^{\infty} a_i$ converges to a finite number, but $\sum_{i=1}^{\infty} |a_i| = \infty$. This is called a *conditionally convergent* series. Then, for any real number r, the sequence can be rearranged so that the series converges to r. (Formally, there exists a bijection $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\lim_{n\to\infty} \sum_{i=1}^{n} a_{\sigma(i)} = r$.)

2 Problems

1. Which of these these series converge or diverge: $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{2n}$, $\sum_{n=10}^{\infty} \frac{1}{n \log n}$, $\sum_{n=100}^{\infty} \frac{1}{n \log n \log\log n}$, $\ldots$?

2. Prove that the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$ is satisfied by $x = \sqrt{2}$, but that the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4$ has no solution. What is the "break-point"?

3. The expression $\sqrt{-\frac{2}{9} + \sqrt{-\frac{2}{9} + \sqrt{-\frac{2}{9} + \cdots}}}$ is not clearly defined. Consider the recursion $a_{n+1} = \sqrt{-\frac{2}{9} + a_n}$ with $a_1 = 0.34$. What does that sequence converge to, and why?

4. Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers such that for each k, a_k is either $\frac{1}{k}$ or $-\frac{1}{k}$, and a_k has the same sign as a_{k+8}. Show that if four of $a_1, a_2, \ldots, a_8$ are positive, then $\sum_{k=1}^{\infty} a_k$ converges. Is the converse true?

5. Prove that a sequence of positive numbers, each of which is less than the average of the previous two, is convergent.

6. Prove that every sequence of real numbers contains a monotone subsequence. Formally, show that for every sequence $a_1, a_2, \ldots$ of real numbers, there is a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(i) < f(j)$ for all $i < j$, and either $a_{f(1)} \le a_{f(2)} \le a_{f(3)} \le \cdots$ or $a_{f(1)} \ge a_{f(2)} \ge a_{f(3)} \ge \cdots$.

7. Given a convergent series of positive terms, $\sum_{k=1}^{\infty} a_k$, does the series $\sum_{k=1}^{\infty} \frac{a_1 + \cdots + a_k}{k}$ always converge? How about the series $\sum_{k=1}^{\infty} \sqrt[k]{a_1 \times \cdots \times a_k}$?

8. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_n < a_{n+1} + a_{n^2}$ for all n. Prove that $\sum a_n$ diverges.

9. Let a_i be real numbers such that $\sum_1^\infty a_i = 1$ and $|a_1| > |a_2| > |a_3| > \cdots$. Suppose that f is a bijection from the positive integers to itself, and

$$|f(i) - i||a_i| \to 0$$

as $i \to \infty$. Prove or disprove that $\sum_1^\infty a_{f(i)} = 1$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.