6. Inequalities

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1 Well-known statements

AM-GM. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + \cdots + a_n}{n},$$

with equality if and only if all a_i are equal.

Cauchy-Schwarz. Let v and w be vectors in an inner product space. Then

$$|\langle v, w \rangle|^2 \le \langle v, v \rangle \cdot \langle w, w \rangle,$$

with equality only if v and w are proportional. Equivalently, if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are sequences of real numbers, then

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \le \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

Smoothing principle. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then if $x + y = x' + y'$ but x' and y' are closer together, we have

$$f(x') + f(y') \le f(x) + f(y).$$

Furthermore, if f is strictly convex, then the inequality is strict.

Compactness. If D is a compact set and $f : D \to \mathbb{R}$ is continuous, then f achieves a maximum on D, i.e., there is at point $x \in D$ such that for all $y \in D$, $f(x) \ge f(y)$.

Jensen. If f is a convex function, then $f(\text{average of } x\text{'s}) \le \text{average of } f(x)\text{'s}$. This implies, for example, that $x^p y^{1-p} \le px + (1-p)y$.

2 Problems

1. Joe has a higher batting average than Mike for the first half of the season, *and* Joe also has a higher batting average than Mike for the second half of the season. Does it follow that Joe has a higher batting average than Mike for the whole season?

2. If $a_1 + \cdots + a_n = n$, prove that $a_1^4 + \cdots + a_n^4 \ge n$.

3. Let $a_1, \ldots, a_n$ be distinct real numbers. Find the maximum of

$$a_1 a_{\sigma(1)} + a_2 a_{\sigma(2)} + \cdots a_n a_{\sigma(n)},$$

over all permutations of the set $\{1, \ldots, n\}$.

4. At every lattice point in the plane there is placed a positive number in such a way that each is the average of its four nearest neighbors. Show that all the numbers are the same. (A lattice point is a point whose coordinates are both integers.)

5. Prove that in any set of 2000 distinct real numbers there exist four elements a, b, c, d with:

   - $a > b$, and
   - $c > d$, and
   - $a \neq c$ or $b \neq d$,

   such that
   $$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

6. Show that if $f(x)$ is a function from $\mathbb{R} \to \mathbb{R}$, whose first and second derivatives both exist and are continuous, and both $f(x)$ and $f''(x)$ are bounded, then $f'(x)$ is also bounded.

7. Let $a_1, a_2, \ldots, a_n$ be real numbers, and suppose that b is a real number for which
   $$b < \frac{\left(\sum a_i\right)^2}{n-1} - \sum a_i^2.$$
   Show that $b < 2a_i a_j$ for all distinct pairs of i and j.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.