5. Functional equations

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1 Well-known statements

Cauchy. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then there must be a real number c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Discontinuous Cauchy. Without the continuity assumption, there are more solutions (using the Axiom of Choice).

Triple iterate. Let $f(x) = 1 - \frac{1}{x}$. Then $f(f(f(x))) = x$.

2 Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in \mathbb{R}$. Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

2. Determine all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ which satisfy

$$f(xy) = f(x) + f(y)$$

for all positive real numbers x, y.

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y) = f(x) + f(y) + f(x)f(y).$$

4. Let n_1, n_2, n_3, ... be a sequence of positive integers with the property that for every $k \geq 1$,

$$n_{k+1} > n_{n_k}.$$

Prove that this must be the sequence $1, 2, 3, \ldots$

5. Do there exist continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all real numbers x?

6. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all real numbers x?

7. There is a function $f(x)$, continuous on the whole real line, which is not identically zero, but satisfies the equation

$$f(x) + f(2x) + f(3x) = 0$$

for all $x \in \mathbb{R}$.

8. Define the recursion:

$$\begin{aligned}\ell_0(s) &= e^{-s} \\ \ell_{t+1}(s) &= \frac{1}{1+\ell_t(\frac{1}{2})}\left[\begin{array}{l} \ell_t\left(s\cdot\frac{1+\ell_t(\frac{1}{2})}{2}\right)^2 - \ell_t\left(s\cdot\frac{1+\ell_t(\frac{1}{2})}{2}\right)\ell_t\left(\frac{1}{2}+s\cdot\frac{1+\ell_t(\frac{1}{2})}{2}\right) \\ +\ell_t\left(\frac{1}{2}+s\cdot\frac{1+\ell_t(\frac{1}{2})}{2}\right)+\ell_t\left(\frac{1}{2}\right)\ell_t\left(s\cdot\frac{1+\ell_t(\frac{1}{2})}{2}\right) \end{array}\right]\end{aligned}$$

Prove that $\ell_t(\frac{1}{2}) \to 1$ as $t \to \infty$.[1]

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.

[1] P. Loh and E. Lubetzky, Stochastic coalescence in logarithmic time, *Annals of Applied Probability*, **23** (2013), 492–528.