4. Calculus

Po-Shen Loh

CMU Putnam Seminar, Fall 2022

1 Well-known statements

Gaussian. $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$.

Archimedes' Principle. If you take a (perfectly spherical) orange, and slice it with a bagel slicer (with blades 2 cm apart), where both blades cut the orange, the surface area of peel you obtain is exactly the same no matter where along the orange you slice.

Volume of torus. The volume of a torus is $(\pi r^2)(2\pi R)$, where r is the radius of the circular cross section, and R is the distance from the center of the torus to the center of a circular cross section.

2 Problems

1. Determine $f'(x)$, if $f(x) = \left[\int_0^{x^2} e^{-t^2} dt\right]^2$.

2. Let C be the unit circle $x^2 + y^2 = 1$. A point P is chosen randomly on the circumference C and another point Q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal PQ. What is the probability that no point of R lies outside of C?

3. Find all real functions f for which $\int_0^x f(t)dt = \frac{1}{2} x f(x)$.

4. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left| \int_0^{\alpha} f(x)dx \right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)|.$$

5. Find the volume of the region of points (x, y, z) such that
$$(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2).$$

6. Let P be a convex polygon, let Q be the interior of P, and let $S = P \cup Q$. Let p be the perimeter of P and let A be its area. Given any point (x, y), let $d(x, y)$ be the distance from (x, y) to the nearest point of S. Find constants α, β, and γ such that
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-d(x,y)} dx dy = \alpha + \beta p + \gamma A.$$

7. Let G_n be the geometric mean of $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Calculate:
$$\lim_{n\to\infty} \sqrt[n]{G_n}.$$

8. Use Fourier series (or any other method) to prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

9. Using the Fourier series of $|x|$, prove that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

10. Evaluate

$$\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt.$$

11. Let V be the pyramidal region $x, y, z \geq 0$, $x + y + z \leq 1$. Evaluate

$$\int_V xy^9 z^8 (1 - x - y - z)^4 dxdydz.$$

12. Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ such that (i) for every $x > 0$, $f(x) > 0$, and (ii) for all $a > 0$, the centroid of the region under the curve $y = f(x)$ between $0 \leq x \leq a$ has y-coordinate equal to the average value of $f(x)$ on $[0, a]$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.