3. Number theory

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CMU Putnam Seminar, Fall 2022

1 Well-known statements

Fermat's Little Theorem. For every prime p and any integer a which is not divisible by p, we have $a^{p-1} \equiv 1 \pmod{p}$.

Euler's Theorem. Let $\varphi(n)$ denote the number of positive integers in $\{1, 2, \ldots, n\}$ which are relatively prime to n. Then, for any integer a which is relatively prime to n,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Wilson's Theorem. A positive integer n is a prime if and only if $(n-1)! \equiv -1 \pmod{n}$.

Dirichlet's Theorem. For any two positive integers a and d which are relatively prime, the arithmetic progression $a, a+d, a+2d, \ldots$ contains infinitely many primes.

Quadratic residues. Let p be a prime. There are exactly $\frac{p+1}{2}$ residues r such that there exist solutions to $x^2 \equiv r \pmod{p}$.

2 Problems

1. The 9-digit number 2^{29} has exactly 9 digits, and they are all distinct. Which of the 10 possible digits 0–9 does not appear?
2. There are infinitely many primes of the form $4n-1$, where n is an integer.
3. Let p be a prime, and let $n \geq k$ be non-negative integers. Prove that
$$\binom{pn}{pk} \equiv \binom{n}{k} \pmod{p}.$$
4. Show that for every positive integer n, there is an integer $N > n$ such that the number 5^n appears as the last few digits of 5^N. For example, if $n = 3$, we have $5^3 = 125$, and $5^5 = 3125$, so $N = 5$ would work.
5. Prove that the product of 3 consecutive integers is never a perfect power (i.e., a perfect square, a perfect cube, etc).
6. How many integers r in $\{0, 1, \ldots, 2^n - 1\}$ are there for which there exists an x where $x^2 \equiv r \pmod{2^n}$?
7. Let n, a, b be positive integers. Prove that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.
8. A positive integer is wrtten at each integer point in the plane ($\mathbb{Z}^2$), in such a way that each of these numbers is the arithmetic mean of its four neighbors. Prove that all of the numbers are equal.
9. A triangular number is a positive integer of the form $n(n+1)/2$. Prove that m is the sum of two triangular numbers if and only if $4m+1$ is the sum of two squares.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.