# 1. Introduction

Po-Shen Loh

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## 1 Well-known statements

**Well-ordering principle.** Every non-empty set of positive integers contains a minimum element.

**Place value.** There are 6 buckets in a row, with one penny in each bucket. You have one magic move, which you can perform as many times as you wish: you may remove one penny from a bucket, and add two pennies to the bucket to its immediate right. (If you removed a penny from the rightmost bucket, no pennies are added.) You may stop at any time. How many pennies can you end up with in total over the 6 buckets?

**Fermat's Last Theorem for $n = 4$.** The equation $a^4 + b^4 = c^4$ has no positive integer solutions.

## 2 Problems

1. There are 6 buckets in a row, with one penny in each bucket. You have two magic moves, which you may perform as many times as you wish. Move A allows you to remove one penny from a bucket, and add two pennies to the bucket to its immediate right. (If you removed a penny from the rightmost bucket, no pennies are added.) Move B allows you to remove one penny from a bucket, and swap the contents of the two buckets to that bucket's immediate right. (If you removed a penny from the rightmost bucket or the second-rightmost bucket, then no swapping happens.) You may stop at any time. How many pennies can you end up with in total over the 6 buckets?

2. Prove that the number of pennies in the problem above cannot go to infinity.

3. Start with a square with side length 5. Place a 3-4-5 right triangle on the outside border of the square, where the hypotenuse is one side of the square. On the opposite side of the square, place another 3-4-5 right triangle in such a way that its hypotenuse is that side of the square, and the rest of the triangle is also outside the square, but orient the triangles so that if the whole diagram is rotated 180 degrees, it looks the same as it started. What is the distance between the two right-angled corners on the two 3-4-5 triangles?

4. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite increasing sequence of positive integers. Prove that there is one and exactly one integer $n \geq 1$ such that
$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leq z_{n+1}.$$

5. Show that $(36x + y)(x + 36y)$ is not a power of 2 for any positive integers $x$ and $y$.

6. Calculate $\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1}$.

7. Let $n > 1$ be an odd integer, and let $\omega = e^{\pi i/n}$. Find integers $a_0, a_1, \ldots, a_n$ such that $\sum a_k \omega^k = \frac{1}{1-\omega}$.

8. Prove that $\cos^{-1}\frac{1}{3}$ is an irrational multiple of $\pi$. (We are working in radians.)

9. Let $S$ be the set of all triples $(x, y, z)$ of positive irrational numbers (not necessarily distinct) such that $x + y + z = 1$. Given a triple $P = (x, y, z) \in S$, let $P' = (\{2x\}, \{2y\}, \{2z\})$, where $\{t\}$ denotes the fractional part of $t$, i.e., $\{1.258\} = 0.258$. Does repeated application of this operation necessarily produce a triple with all elements $< \frac{1}{2}$, no matter what triple from $S$ one starts with?

## 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.