Putnam Σ.9

Po-Shen Loh

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1 Problems

There is 82% chance of rain during dinner, so we'll go together to Orient Express and order once we get there.

Putnam 2015/A4. For each real number x, let

$$f(x) = \sum_{n \in S_x} \frac{1}{2^n},$$

where S_x is the set of positive integers n for which $\lfloor nx \rfloor$ is even. What is the largest real number L such that $f(x) \geq L$ for all $x \in [0, 1)$? (As usual, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Putnam 2015/A5. Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

Putnam 2015/A6. Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.