Putnam Σ.3

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1 Problems

Putnam 2004/B4. Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

Putnam 2004/B5. Evaluate
$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

Putnam 2004/B6. Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then
$$\lim_{x \to \infty} N(x)/x = 0.$$