8. Recursions

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1 Famous results

Monotone sequences. A sequence $a_1, a_2, \ldots$ is called *monotone increasing* if $a_{n+1} \geq a_n$ for all n, and *monotone decreasing* if $a_{n+1} \leq a_n$ for all n. A *monotone sequence* refers to a sequence that is in one of these two categories. Every monotone sequence converges to a limit.

Wallis product. Wow!

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2} \,.$$

2 Problems

1. Define the sequence $a_0 = -1$, $a_1 = 0$, and
$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1 \,.$$
Find a_{100}.

2. For a real number α, define the sequence $a_1, a_2, a_3, \ldots$ by starting $a_1 = \alpha$, and defining $a_{n+1} = a_n(1 - a_n)$ for every $n \geq 1$. Prove that if $0 < \alpha < 1$, the resulting sequence always converges to 0. What happens if $\alpha < 0$? What happens if $\alpha > 1$? What happens if $\alpha = 0$ or $\alpha = 1$?

3. Let $a_0 = 0$, and for each $n \geq 0$, let $a_{n+1} = 1 + \sin(a_n - 1)$. Determine $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n a_i$.

4. For a real number α, define the sequence $a_1, a_2, a_3, \ldots$ by starting $a_1 = \alpha$, and defining $a_{n+1} = a_n(1 - a_n)$ for every $n \geq 1$. Prove that if $0 < \alpha < 1$,
$$\lim_{n\to\infty} n a_n = 1 \,.$$

5. Let $a_1, a_2, a_3, \ldots$ be a sequence which satisfies $a_1 a_2 = 1$, $a_2 a_3 = 2$, $a_3 a_4 = 3$, $a_4 a_5 = 4$, etc., as well as $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$. Prove that $a_1 = \sqrt{\frac{2}{\pi}}$.

6. Let $f : [0,1] \to [0,1]$ be a continuous function. Let $x_1, x_2, \ldots$ be a sequence satisfying $x_{n+1} = f(x_n)$ for all $n \geq 1$, and suppose that
$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0 \,.$$
Prove that the entire sequence $x_1, x_2, \ldots$ converges.

7. Let
$$a_n = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\cdots + (n-3)\sqrt{1 + (n-2)\sqrt{1 + (n-1)\sqrt{1 + (n)}}}}}}} \,.$$
Prove that $\lim_{n\to\infty} a_n = 3$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.