# 6. Inequalities

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CMU Putnam Seminar, Fall 2021

## 1 Famous results

**Cauchy-Schwarz.** Let $v$ and $w$ be vectors in an inner product space. Then

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle ,$$

with equality only if $v$ and $w$ are proportional.

**Jensen.** If $f$ is a convex function, then $f$(average of $x$'s) $\leq$ average of $f(x)$'s. This implies, for example, that $x^p y^{1-p} \leq px + (1-p)y$.

**Bieberbach, via Steiner symmetrization.** Every compact set $K \subset \mathbb{R}^n$ satisfies

$$\mathrm{vol}(K) \leq \mathrm{vol}(B_n) \cdot \left(\frac{\mathrm{diam}(K)}{2}\right)^n ,$$

where $B_n$ is the unit ball in $n$ dimensions, and $\mathrm{diam}(K) = \max\{\mathrm{dist}(x, y) : x, y \in K\}$ is the diameter of set $K$.

**Beats.** Superpositions of sine waves can form "beats":

$$\frac{1}{2}\sin a\theta + \frac{1}{2}\sin b\theta = \sin\left(\frac{a+b}{2}\theta\right)\cos\left(\frac{a-b}{2}\theta\right) .$$

## 2 Problems

1. In terms of $n$, determine the maximum possible value of the sum

$$\sum_{1\leq i<j\leq n} |x_i - x_j|$$

where $x_1, \ldots, x_n$ are (not necessarily distinct) real numbers in $[0, 1]$.

2. Determine the minimum possible value of the sum

$$\sum_{1\leq i<j\leq n} |x_i - x_j|$$

where $x_1, \ldots, x_n$ are (not necessarily distinct) real numbers in $[0, 1]$, and one of them is 0 and one of them is 1.

3. Find all pairs of real numbers $(\alpha, \beta)$ for which there is a constant $C$ such that for all positive reals $x$ and $y$,

$$x^\alpha y^\beta < C(x + y) .$$

4. Let $\alpha$ be a real number. Are there any continuous real-valued functions $f : [0, 1] \to \mathbb{R}^+$ such that

$$\int_0^1 f(x)dx = 1\,, \qquad \int_0^1 xf(x)dx = \alpha\,, \text{and} \qquad \int_0^1 x^2 f(x)dx = \alpha^2\,?$$

5. Suppose that $a_1, \ldots, a_n$ are real numbers such that

$$\left| \sum_{k=1}^{n} a_k \sin kx \right| \leq |\sin x|$$

for all real $x$. Prove that

$$\left| \sum_{k=1}^{n} k a_k \right| \leq 1\,.$$

6. Let $K$ be a convex set in the plane with area at least $\pi$, whose boundary is a finite collection of line segments. Prove that there are points $X, Y \in K$ such that the distance between $X$ and $Y$ is at least 2.

7. Let $P_1, P_2, \ldots, P_n$ be points on the surface of the unit sphere in $\mathbb{R}^3$, i.e., with coordinates satisfying $x^2 + y^2 + z^2 = 1$. Prove that

$$\sum_{1 \leq i < j \leq n} d_2(P_i, P_j)^2 \leq n^2\,,$$

where $d_2(X, Y)$ is the ordinary Euclidean distance between points $X$ and $Y$.

8. Show that for every curve in $\mathbb{R}^2$ of length 1, there is a closed rectangle of area $\frac{1}{4}$ which covers it completely.

9. Show that a circle inscribed in a square has a larger perimeter than any other ellipse inscribed in a square.

# 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.