1. Introduction

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1 Famous results

Newton's binomial theorem. For real numbers r and positive integers k, let $\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$. Then, for any $r \in \mathbb{R}$ and any real x with $|x| < 1$,

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k .$$

Triangulation. Every 2-dimensional polygon (convex or non-convex) can be decomposed into triangles, where all vertices of the triangles were also vertices of the original polygon, and triangles overlap only along edges.

Simplicial decomposition. Every convex polytope (convex hull of a finite point set in $\mathbb{R}^d$) admits a decomposition into d-dimensional simplices, where all vertices of the simplices are also vertices of the original polytope.

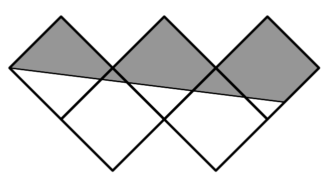
Schönhardt. There are non-convex 3-dimensional polytopes which cannot be decomposed into tetrahedra.

2 Problems

1. On a standard die, opposite faces sum to 7. Why?

2. A *lattice point* is a point with all-integer coordinates. Prove that given any 9 distinct lattice points A, B, …, I in 3-dimensional space, some two of them have the property that the straight line segment between them also contains another lattice point. (This lattice point does not have to be one of the given A, …, I.)

3. This W is cut into two pieces of equal area by the straight line. The straight line intersects the right edge of the W at a point. Exactly what ratio does this point split the lengths of the right edge into?



4. There are 2013 lockers in a gym, numbered from 1 to 2013, and they all start closed. There are also 2013 students. On the first day, student #1 toggles the status of lockers 1, 2, 3, …, 2013. (Since they were all closed, they are now all open.) On the second day, student #2 toggles the status of lockers 2, 4, 6, …, 2012. On the i-th day, student i toggles the status of lockers i, $2i$, $3i$, …. How many lockers are open after all 2013 students have passed through?

5. Given a set of points $S \in \mathbb{R}^2$, let $f(S)$ denote the point set

$$\bigcup_{A,B \in S} \{\text{all points on the line segment } AB \text{, including endpoints}\}.$$

Prove that for every set $S \in \mathbb{R}^2$, $f(f(S)) = f(f(f(S)))$.

6. (Harder.) Prove that this also works for point sets $S \in \mathbb{R}^3$.

7. (Binomial theorem for falling factorials.) For any positive integer n and any real number x, let the *falling factorial* $(x)_n$ be the product of n numbers $x(x-1)(x-2)\cdots(x-n+1)$. Prove that

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}\,.$$

This also holds for rising factorials $x^{(n)} = x(x+1)\cdots(x+n-1)$.

8. Let $\alpha > \frac{1}{2}$ be a real number. Prove that it is impossible to find a real function f such that

$$f(x) = 1 + \alpha \int_x^1 f(t) f(t-x) dt$$

holds for all $0 \le x \le 1$.

9. Let S be a set of $mn+1$ distinct positive integers. Prove that there must always exist (1) a sequence of $m+1$ distinct integers $a_1, a_2, \ldots, a_{m+1} \in S$ for which $a_1 \mid a_2 \mid \ldots \mid a_{m+1}$, or (2) a subset of $n+1$ distinct integers from S, none of which divides another.

10. (Gallai, Hasse, Roy, Vitaver.) Let D be a directed graph, and let χ be the chromatic number of its underlying undirected graph. Show that D has a directed path of at least χ vertices.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.