21-228 Discrete Mathematics

Assignment 2

Due Fri Feb 19, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. It is well-known that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2},$$
$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \text{ and}$$
$$\sum_{k=0}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2,$$

but it is hard to remember the formulas for sums of higher powers. On the other hand, the following formula is true:

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{1}{4}(n+1)(n)(n-1)(n-2). \tag{1}$$

This is easier to remember, because it looks like $\int x^3 dx = \frac{1}{4}x^4$. You don't need to write up proofs of any of the above statements (although it might be good practice to do so). For this problem, discover and prove a general formula for sums of the following form:

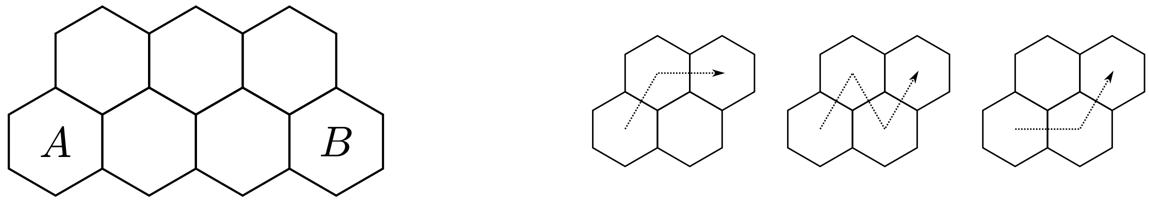
$$\sum_{k=0}^{n} k(k-1)(k-2)\cdots(k-r+1).$$

Note that there are r factors in this product, and (1) corresponds to the $r = 3$ case. Your formula should be in terms of n and r.

2. Consider beehives whose hexagonal cells appear in only two rows. Some bees would like to figure out how many ways there are to move from the leftmost cell to the rightmost cell, using only moves of three types: to the east, to the northeast, and to the southeast. For each positive integer, let a_n denote the number of different ways when there are exactly n cells (arranged in the two-row pattern as shown in Figure 1). If n is odd, the bottom row will have one more cell than the top row, and if n is even, both rows will have the same number of cells. For example, $a_1 = 1$ and $a_4 = 3$.

Calculate some small values of a_n to gain intuition about this sequence (you don't need to submit this work), and then prove that for all positive integers n:

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1.$$



(a) (b)

Figure 1: The case $n = 7$ is illustrated in part (a), and and the bees would like to count the number of paths from A to B. Part (b) shows the three paths for $n = 4$, corresponding to $a_4 = 3$.

3. Three couples want to take some pictures, seated in a row. (There are 6 people, in three pairs.) How many ways can they do this, with the restriction that no person is right next to his/her partner? For example, if the couples are (Alice, Bob), (Claire, David), and (Ernest, Felicia), then one possible arrangement is ACEBDF, but CABEDF does not count.

4. A set S of numbers is called a *Sidon set* if it has the property that for every distinct $a, b, c, d \in S$, the sums $a + b$ and $c + d$ are different. For example, $\{1, 2, 4, 8\}$ is Sidon, but $\{1, 2, 3, 4\}$ is not because $1 + 4 = 2 + 3$. A natural question is to ask how large a Sidon set can be, if, say, the numbers must be integers in $\{1, \ldots, 100\}$. Prove that no Sidon set with all elements in $\{1, \ldots, 100\}$ can have size more than 20 (i.e., 21 or more).

   For partial or extra credit: A proof that every Sidon subset of $\{1, \ldots, 100\}$ must have size $\leq N$ will receive $30 - N$ points.

5. For every positive integer r, let N_r denote the minimum integer such that the following holds: whenever the squares in an $N_r \times N_r$ chessboard are colored in r colors, there are always 4 squares which have the same color, and form the corners of a rectangle. In recitation, we saw that $N_r \leq r^{r+1} + 1$. Prove a polynomial bound on N_r: i.e., show that there are constants C and c such that $N_r \leq Cr^c$ for all positive integers r.