21-228 Discrete Mathematics (2021)

Final Exam

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Name: ____________________________________

This 3-hour exam is open-notes, in the sense that you may use anything you have written yourself. No calculators are permitted. Please write your answers in the space provided, and indicate clearly on the front of a page if you use the back of that page for additional space. Every numerical answer must be justified with an explanation. You may use any theorem which was stated in class without reproving it. Each problem is worth 10 points.

Note: questions are arranged roughly in order of difficulty.

1. You are given a graph. It has 100 vertices, exactly 5 connected components, and no odd-length cycles. How many ways are there to color all of the vertices, where each vertex can be either black or white, and no edge has both endpoints in the same color?

2. Suppose that $a_n = a_{n-1} + 2a_{n-2}$, and $a_1 = 0$ and $a_2 = 6$. Find and justify a formula for a_n which works for all positive integers n. Your formula may not use summation notation (Σ), or product notation (Π), or ellipses ($\cdots$).

3. There are 30 students in a class. How many ways are there to split them up into 10 groups of 3 students each, to work together for a project? The groups don't have numbers or names of their own, and from the point of view of everyone, all that matters is who is with who in each group. You may express your answer as an arithmetic expression, which may use factorials, but it should not use summation notation (Σ) or product notation (Π) or ellipses ($\cdots$).

4. The expansion of $(a+b+c)^2$ has 6 terms after collecting like terms:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

How many terms are there in the expansion of $(a+b+c)^8$ after collecting like terms? You may express your answer as an arithmetic expression, which may use factorials, but it should not use summation notation (Σ) or product notation (Π) or ellipses ($\cdots$).

5. Show that the number of integers in $\{1, 2, \ldots, 420\}$ which are not multiples of any of 2, 3, or 7 is

$$420\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{7}\right)=120.$$

You may use facts like "there are exactly $\frac{420}{6}$ integers in $\{1, \ldots, 420\}$ which are multiples of both 2 and 3, because those are just multiples of 6, and 420 is divisible by 6." You may not use fancy facts like the Chinese Remainder Theorem or Euler's Totient Function, because not everyone in this class knows those facts. Inclusion-Exclusion might be useful.

6. Suppose that someone gives you a bipartite graph, and they promise that it contains a perfect matching from left to right. You have found a not-perfect partial matching (so not every vertex on the left is matched yet). Prove that there always exists an augmenting path which starts from a vertex on the left which is not used in your partial matching, and ends at a vertex on the right which is not used in your partial matching. You may use any theorem stated in class, but if you wish to use or adapt any parts of the proof, you must explain why those steps are true. (It's not permitted to use "just like the proof of Theorem X.") This problem is simpler than it looks.

7. Suppose that $a_0 = -1$, and

$$\begin{aligned} a_1 &= a_0 a_0 \\ a_2 &= a_0 a_1 + a_1 a_0 \\ a_3 &= a_0 a_2 + a_1 a_1 + a_2 a_0, \end{aligned}$$

and more generally, for each positive integer n:

$$a_{n+1} = a_0 a_n + a_1 a_{n-1} + \cdots + a_{n-1} a_1 + a_n a_0.$$

Find and justify a formula for a_n which works for all positive integers n. Your formula may use factorials, but it may not use summation notation (Σ) or product notation (Π) or ellipses ($\cdots$).

8. Let B and W be positive integers with $B \geq W$. How many ways are there to arrange B black marbles and W white marbles in a line so that no white marbles are right next to each other? Assume the black marbles are indistinguishable from each other, and also assume that the white marbles are indistinguishable from each other. You may express your answer in terms of factorials, but you may not use summation notation (Σ), product notation (Π), or ellipses ($\cdots$).