21-228 Discrete Mathematics (2021)

Final Exam Alternate Time

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Name: ____

This 3-hour exam is open-notes, in the sense that you may use anything you have written yourself. No calculators are permitted. Please write your answers in the space provided, and indicate clearly on the front of a page if you use the back of that page for additional space. Every numerical answer must be justified with an explanation. You may use any theorem which was stated in class without reproving it. Each problem is worth 10 points.

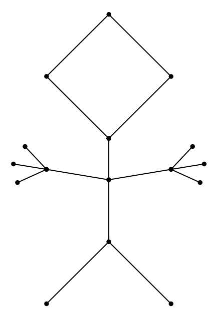
Note: questions are arranged roughly in order of difficulty.

1. Suppose that a graph has 100 vertices, and each vertex is part of some cycle. (The same cycle can fulfill this property for multiple vertices.) Prove that the graph has at least 100 edges.

2. Find two numbers α and β such that the recursion $a_n = \alpha a_{n-1} + \beta a_{n-2}$ is satisfied by both of the sequences $a_n = 2^n$ and $a_n = 3^n$ (but with different initial conditions).

3. How many different ways are there to label the 16 vertices of this diagram using the distinct numbers $\{1, 2, \ldots, 16\}$, where "different" means that when they're thought of as labeled graphs, they're different labeled graphs? Two labeled graphs are the same if they have exactly the same pairs of vertices being adjacent. For example, if you take one labeling, and just swap the labels on two fingertips on the same hand, then that gives the same labeled graph, and you shouldn't count it again. On the other hand, if you swap the labels on the chin and a foot, then that gives a different labeled graph, and you should count those two separately.

   You may express your answer as an arithmetic expression, which may use factorials, but it should not use summation notation (Σ) or product notation (Π) or ellipses ($\cdots$).



4. What is the sum of all of the Binomial coefficients where both the top and bottom number are odd positive integers less than 100? Your answer may be in the form of an arithmetic expression, but you may not use summation notation (Σ) or product notation (Π) or ellipses ($\cdots$).

5. An election just finished between Alice and Bob. Each of them got exactly 100 votes, but nobody knows that yet. The 200 ballots are strewn on a table, of which 100 are for Alice and 100 are for Bob. When they are counted, they're counted one-by-one in a uniformly random sequential order (so any of the 200! possible ballot orderings are equally likely). What is the probability that over the course of the counting, there is both at least some moment when Alice is ahead in the partial count, and at least some moment when Bob is ahead in the partial count?

6. What is the maximum number of edges in any 8-vertex planar graph which has no 3 pairwise-adjacent vertices (no triangle)? To complete this question, you need to state a number E, and both prove that every such graph has less than or equal to E edges, and also draw an example of an 8-vertex planar graph with exactly E edges and no triangles.

7. How many multiples of 3 are there in this list?

$$\binom{3}{3}, \binom{4}{3}, \binom{5}{3}, \binom{6}{3}, \ldots, \binom{81}{3}?$$

8. For each integer $n \geq 3$, find and justify a formula for the number of ways to properly color the vertices of a labeled n-vertex cycle, where each vertex is either red, green, or blue. A proper coloring is one in which no edge has both endpoints in the same color. It is OK for a proper coloring to use only 2 of the colors.

   Since the cycle is labeled, there would be 3^n ways to color the vertices, but some of those colorings are bad because they have edge(s) with both endpoints of the same color. Don't count the bad ones.

   Hint: the number of ways to properly color an n-vertex cycle equals the number of ways to properly color an n-vertex path, minus those ways where the first and last vertices happen to be the same color. That number you are subtracting has some relation to an $(n-1)$-vertex cycle.