21-228 Discrete Mathematics Exam 3

April 30, 2021

Problem	**Score**
1	
2	
3	
4	
5	
Total	

Name: ______________________________

This 50-minute exam is open-notes, in the sense that you may use anything you have written yourself. No calculators are permitted. Please write your answers in the space provided, and indicate clearly on the front of a page if you use the back of that page for additional space. Every numerical answer must be justified with an explanation. You may use any theorem which was stated in class without reproving it. Each problem is worth 10 points.

1. There is a graph in which all vertices have degree exactly 3, and there are no cycles with 6 or fewer vertices. Consider an arbitrary vertex v in the graph. How many vertices are distance exactly 3 from v? (The distance between two vertices is the number of edges in the shortest path between them, or infinity if the two vertices are not connected by any path.)

2. Show that every 30-vertex tree with exactly 25 leaves and exactly 5 non-leaves contains a vertex of degree at least 7.

3. How many n-vertex trees with vertices labeled $\{1, 2, \ldots, n\}$ are there where the leaf with the **largest** label is adjacent to the vertex labeled n? (Note that then n is definitely not a leaf.) Your answer should not use summation notation (Σ), product notation (Π), or ellipses ($\cdots$).

4. Suppose that S is a set of lattice points in the plane. These are points (x, y) where both x and y are integers. Suppose that for every point in S, there are at least 2 other points in S which are distance exactly 1 away (according to the usual definition of distance in the plane). Show that there is an sequence of distinct points in S where the number of points is an even number greater than or equal to 4, such that every pair of consecutive points in the sequence is at distance exactly 1 apart, and the first and last points in the sequence are also at distance exactly 1 apart.

5. Let the graph G be the complete bipartite graph $K_{10^{100},10^{100}}$, which means that it has 10^{100} vertices on the left, 10^{100} vertices on the right, and the set of 10^{200} edges is precisely all edges which cross between the left and the right. Prove that no matter how the 10^{200} edges are colored red or blue, there is a monochromatic cycle with 4 edges.