21-228 Discrete Mathematics Exam 2

March 26, 2021

Problem	**Score**
1	
2	
3	
4	
5	
Total	

Name: ______________________________

This 50-minute exam is open-notes, in the sense that you may use anything you have written yourself. No calculators are permitted. Please write your answers in the space provided, and indicate clearly on the front of a page if you use the back of that page for additional space. Every numerical answer must be justified with an explanation. You may use any theorem which was stated in class without reproving it. Each problem is worth 10 points.

1. Consider the recursion $a_n = a_{n-1} + 6a_{n-2}$ with initial conditions $a_1 = 1$ and $a_2 = 13$. Determine a formula for a_n in terms of n. Your formula may use standard arithmetic operations, but may not use summation notation (Σ) or ellipses ($\cdots$).

2. Find numbers A, B, and C such that $a_n = n2^n + 2$ is a solution to the recursion $a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$.

3. Find numbers A and B, and initial values a_0 and a_1 such that the recursion $a_n = Aa_{n-1} + Ba_{n-2}$ with those initial values has generating function

$$a_0 + a_1 z + a_2 z^2 + \cdots = \frac{1}{1 + z + z^2}.$$

4. How many paths are there from $(0,0)$ to $(n-2, n)$, which only take steps of integer length, directly upward or directly to the right, where all points (x, y) along the path satisfy $y \geq x$? (This means that the path is always on or above the diagonal line $y = x$.) Your answer may use standard arithmetic operations, factorials, or binomial coefficients, but may not use summation notation (Σ) or ellipses $(\cdots)$.

5. Find a nonzero real number α with the property that for positive integers n, the quantity $\alpha(2+\sqrt{3})^n$ magically ends up being closer and closer to an integer as n increases. (For different values of n, the closest integer is different.) There is more than one possible answer for α, but you just need to find one that works, and justify why.