# 8. Recursions

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## 1 Classical results

**Classical.** Prove that the sequence $\sqrt{7}$, $\sqrt{7+\sqrt{7}}$, $\sqrt{7+\sqrt{7+\sqrt{7}}}$, ... converges, and determine its limit. This is often denoted as $\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$.

## 2 Problems

1. Let $a_1, a_2, \ldots$ be a sequence of real numbers which satisfies $a_{n+1} = \frac{1}{2-a_n}$. Prove that $\lim_{n\to\infty} a_n = 1$.
2. Let $\alpha$ be an arbitrary real number. Define $a_1 = \alpha$, and for all $n \geq 1$, let $a_{n+1} = \cos a_n$. Prove that $a_n$ converges to a limit, and that this limit does not depend on $\alpha$.
3. Let $t_1, t_2, \ldots$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1}^2 = 1 + t_n$, for $n \geq 1$. Show that $t_n$ is increasing in $n$ and find $\lim_{n\to\infty} t_n$.
4. Prove that the sequence $\sqrt{7}$, $\sqrt{7-\sqrt{7}}$, $\sqrt{7-\sqrt{7+\sqrt{7}}}$, $\sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}$, ..., converges, and determine its limit.
5. The sequence $a_n$ is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that any pair of values in the sequence are relatively prime and that $\sum \frac{1}{a_n} = 1$.
6. Define $a_1 = 1$, and let $a_{n+1} = 1 + \frac{n}{a_n}$ for all $n$. Show that $\sqrt{n} \leq a_n < 1 + \sqrt{n}$.
7. Let $a_i$ be a sequence of positive real numbers. Show that $\limsup \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq e$.

## 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.