6. Inequalities

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1 Classical results

Smoothing principle. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then if $x + y = x' + y'$ but x' and y' are closer together, we have

$$f(x') + f(y') \le f(x) + f(y) .$$

Furthermore, if f is strictly convex, then the inequality is strict.

Jensen. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$,

$$f\left(\frac{a_1 + \cdots + a_n}{n}\right) \le \frac{f(a_1) + \cdots + f(a_n)}{n} .$$

Compactness. If D is a compact set and $f : D \to \mathbb{R}$ is continuous, then f achieves a maximum on D, i.e., there is at point $x \in D$ such that for all $y \in D$, $f(x) \ge f(y)$.

AM-GM. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + \cdots + a_n}{n} ,$$

with equality if and only if all a_i are equal.

Cauchy-Schwarz. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) ,$$

with equality only if the sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

Dirichlet approximation. For any real number r and any positive integer N, there are integers a and b with $1 \le b \le N$ which satisfy

$$\left| r - \frac{a}{b} \right| < \frac{1}{b^2} .$$

2 Problems

1. Let P_1, P_2, $\ldots$, P_n be points on a line, not necessarily distinct. Which points P on the line minimize the sum of distances $\sum_i |PP_i|$?

2. Prove that for all positive real numbers a, b, c, the following holds:

$$\frac{9}{a+b+c} \le 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) .$$

3. Given $n > 8$, let $a = \sqrt{n}$ and $b = \sqrt{n+1}$. Which is greater, a^b or b^a?
4. Show that $\log\left(1 + \frac{1}{x}\right) > \frac{1}{1+x}$ for $x > 0$.
5. Let $p(x)$ be a real polynomial of degree at most 2, which satisfies $|p(x)| \le 1$ for all $-1 \le x \le 1$. Show that $|p'(x)| \le 4$ for all $-1 \le x \le 1$.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying $f(0) = 0$ and $|f'(x)| \le |f(x)|$ for all $x \in \mathbb{R}$. Show that f is constant.
7. Let C be a closed plane curve with the property that every pair of points in C are at distance at most 1 apart. Show that we can find a disk of radius $\frac{1}{\sqrt{3}}$ which contains C.
8. Let O be the origin $(0,0)$, and let C be the line segment $\{(x,y) : x \in [1,3], y = 1\}$. Let K be the curve $\{P : \text{for some } Q \in C,\ P \text{ lies on } OQ \text{ and } PQ = 0.01\}$. Let k be the length of the curve K. Is k greater or less than 2?
9. Show that for any rational $0 < \frac{a}{b} < 1$, we have $\left|\frac{a}{b} - \frac{1}{\sqrt{2}}\right| > \frac{1}{4b^2}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.