1. Introduction

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1 Classical results

1. The real number $\sqrt{2}$ is irrational.
2. The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ is irrational.
3. The set of real numbers is uncountable.
4. (Banach-Tarski Paradox.) It is possible to decompose a closed, solid 3-dimensional ball into a finite number of disjoint subjsets, which can then be translated and rotated into new positions such that the resulting set of points is precisely two identical copies of the original ball.

2 Problems

1. You are presented with a 8×8 square board, with some of the squares filled with red, and some white. You can study the board for as long as you wish, after which an opponent will flip exactly one square's color without you looking. When you look back, you are to identify exactly which square flipped in color. What is the minimum number of bits of memory that you need (during the period when you are not looking) in order to guarantee that you can achieve this?
2. X is a subset of the rationals which is closed under addition and multiplication, and it does not contain 0. (That means for any a and b from X, $a+b$ and ab are both in X, including the case where a and b are the same.) For any rational $x \neq 0$, exactly one of x or $-x$ is in X. Show that X is the set of all positive rationals.
3. The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.
4. A, B, C are points of a fixed ellipse E. Show that the area of ABC is maximized if and only if the centroid of ABC is at the center of E. The centroid of a triangle is its center of mass, which also happens to lie at the intersection of its three medians.
5. D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \emptyset$, and $A \cup B = D$. More formally, D is the closed unit disk, including boundary, i.e., all points (x, y) satisfying $x^2 + y^2 \leq 1$. We must show that it is impossible to choose a subset A of D such that there is a geometric transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is a bijection from A to $D \setminus A$. A geometric transformation is a composition of rotations, translations, and reflections.
6. S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

7. There are $n \geq 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Lisa has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then she will clap her hands $n - 1$ times. Every time she claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Lisa wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

   (a) Prove that Lisa can always fulfill her wish if n is odd.

   (b) Prove that Lisa can never fulfill her wish if n is even.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for each $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. (That limit corresponds to sending evaluating $f(\alpha), f(2\alpha), f(3\alpha), \ldots$ and finding the limit of the sequence.) Prove that $\lim_{x\to\infty} f(x) = 0$, where now this limit corresponds to sending x to ∞ along the real axis. That is, for every $\epsilon > 0$, there is a T such that for all real numbers $x > T$, we have $|f(x)| < \epsilon$.

3 Homework

Please write up solutions to two of the problems, to turn in before next week's meeting. One of them may be a problem that we discussed in class.