Putnam E.7

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1 Problems

Putnam 2013/B1. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Putnam 2013/B2. Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

Putnam 2013/B3. Let $\mathcal{P}$ be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in \mathcal{P}$, then $S \cup S' \in \mathcal{P}$ and $S \cap S' \in \mathcal{P}$, and

(ii) if $S \in \mathcal{P}$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in \mathcal{P}$ and T contains exactly one fewer element than S.

Suppose that $f : \mathcal{P} \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in \mathcal{P}.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in \mathcal{P}$?