4. Calculus

Po-Shen Loh

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1 Classical results

Warm-up. Determine $f'(z)$, if

$$f(z) = \int_0^{z^2} e^{-x^2} dx\,.$$

Gaussian. Calculate

$$\int_{-\infty}^{\infty} e^{-x^2} dx\,.$$

Leibniz formula for π. Recursively compute the integral

$$I_n = \int_0^{\pi/4} \tan^{2n} x dx$$

2 Problems

1. Let $f(x) = \int_0^x \sin(t^2 - t + x)dt$. Compute $f''(x) + f(x)$, and deduce that $f^{(12)}(0) + f^{(10)}(0) = 0$. (Here, $f^{(10)}$ indicates the 10th derivative.)

2. Evaluate
$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2\sin x \cos^3 x + 2\sin^2 x \cos^2 x + 2\sin^3 x \cos x} dx.$$

3. Evaluate
$$\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}} dx.$$

4. Three infinitely long circular cylinders, each with unit radius, have their axes along the x, y and z-axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2 + z^2 \leq 1\}$, $\{z^2 + x^2 \leq 1\}$, and $\{x^2 + y^2 \leq 1\}$.)

5. Compute the limit
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$$

6. Evaluate
$$\int_1^2 \frac{\ln x}{2 - 2x + x^2} dx.$$

7. Evaluate
$$\int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x} dx$$
where $0 \leq \arctan(x) < \frac{\pi}{2}$ for $0 \leq x < \infty$.

8. Use the Fourier series of the function of period 1 defined by $f(x) = \frac{1}{2} - x$ for $0 \leq x < 1$ to prove Euler's formula:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.