Putnam Σ.3

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1 Problems

Putnam 2004/A4. Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

Putnam 2004/A5. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Putnam 2004/A6. Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y) dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y) dx\, dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2 \, dx\, dy.$$