12. Probability

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1 Classical results

Elo Rating System. In the Elo Rating System (adapted for many popular games, such as Chess and LoL), players have numerical ratings which update after each game. If player A has rating R_A and player B has rating R_B, then the ratings update as follows after a given game. First, these quantities are defined:

$$E_A = \frac{1}{1 + 10^{(R_B - R_A)/400}}$$
$$E_B = \frac{1}{1 + 10^{(R_A - R_B)/400}}$$

Also, we define S_A and S_B, each of which is either 0 or 1, depending on whether A or B wins. If A wins, then $S_A = 1$ and $S_B = 0$, and if B wins, then $S_B = 1$ and $S_A = 0$. Finally, the ratings are updated to:

$$R'_A = R_A + 25(S_A - E_A)$$
$$R'_B = R_B + 25(S_B - E_B).$$

This has the property that whenever two players play against each other, the difference between the ratings $R_A - R_B$ can be translated into an estimate of the probability that player A beats player B.

Two Envelopes. You are given two envelopes, one with exactly twice as much money as the other. You may open one envelope and look at how much money is inside it, after which you must decide whether to keep that envelope or to switch to the other one. What is the best strategy? (Assume money is infinitely divisible.)

2 Problems

1. You write down two distinct real numbers, one on the left and one on the right. I decide whether to ask you to reveal the left number or the right number. After that, I guess which of your numbers was the larger one. Do I have a strategy for which my chance of winning is strictly greater than half?

2. Baseball player A has a higher batting average than baseball player B in the first half of the season. The same is true in the second half of the season. Who has the higher batting average over the whole season?

3. Oh no! Youre above a pit of fire and the only way out is a 1000-rung escape ladder. Youre on the first rung. Every second, a standard (6 sided) die rolls. If its a 1 or 2, you move down one rung. Anything 3 or greater, you move up one rung. If you move down from the first rung, you fall into the fire! If you move up from the 1000th rung, you escape. What is the probability that you escape, as a percent rounded to the nearest hundredth?

4. You throw a die until you get 6. What is the expected number of throws (including the throw giving 6), conditioned on the event that all throws gave even numbers?

5. Consider a game between two people, each of whom rolls a die. The rules are that the die must roll for a distance of at least 1 foot on the table. The larger number wins the round, but in the event that there is a tie, the roller whose die stopped moving first wins, with one exception: if both are 6's, then they re-roll. What is the optimal strategy?

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.