21-228 Discrete Mathematics

Assignment 3

Due Fri Feb 17, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. How many rearrangements of the word "DOCUMENT" have the three vowels all next to each other? For example, "DOEUCMNT" counts, but not "DOCUEMNT".

2. A word over the alphabet $\{a, b, c, \dots, z\}$ is called *increasing* if its letters appear in alphabetical order. For example, *boost* is increasing, but *hinder* is not. How many increasing words are there of length 52? Count non-English words, so that the answer is not zero. Answers may be expressed in terms of factorials or binomial coefficients, but summation notation and ellipses may not be used.

3. Prove that if we move straight down in Pascal's triangle (visiting every other row), then the numbers we see are increasing.

4. In class, we proved Dirichlet's theorem, which states that for any real number α, there are integers p and q such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}\,.$$
In other words, every real number has a pretty good rational approximation. What if we would like to approximate two different real numbers using the same denominator? Again, it's easy to see that for any real α, β, there are integers p, p' and q such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q} \quad \text{and} \quad \left|\beta - \frac{p'}{q}\right| < \frac{1}{q}\,.$$
In fact, for any real α, β, there are integers p, p', and q such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{3/2}} \quad \text{and} \quad \left|\beta - \frac{p'}{q}\right| < \frac{1}{q^{3/2}}\,.$$
To prove this, show that for any real α, β and any positive integer N, there are integers p and p', and an integer q satisfying $1 \le q \le N^2$, such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN} \quad \text{and} \quad \left|\beta - \frac{p'}{q}\right| < \frac{1}{qN}\,.$$
You may use that for every real numbers x and y satisfying $\{y\} \ge \{x\}$, it is true that $\{y\} - \{x\} = \{y - x\}$. In class, we proved this under the additional assumption that $y > x$, but it's actually true without that assumption.

   Extra credit (10 pts). Use this to solve the following Putnam B6. For each positive real number α, let $S(\alpha)$ denote the set $\{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}$. Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$.

5. ($\star$) In class, we saw a formula which expressed the size of $|A_1 \cup A_2 \cup \ldots \cup A_n|$ in terms of sizes of intersections (e.g., $|A_1 \cap A_2|$, $|A_3|$, etc.), with some coefficients in front (which were ± 1). That counted the number of elements which are in at least one of the sets $A_1, A_2, \ldots, A_n$.

   Determine a formula which computes the number of elements that are in at least **two** of the sets A_1, A_2, $\ldots$, A_n, in terms of the sizes of the intersections $|A_1 \cap A_2|$, $|A_3|$, etc., possibly with some coefficients in front (not necessarily just ± 1).