13.

(Just do it)

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1 Classical results

Information. Cody, Joe, and Kumail are playing a game. Cody starts in a different room, and Joe and Kumail start in the same room. Kumail starts by putting 64 pennies on a chessboard. He can make some of them heads and some tails, in any way that he wants. He then points to one of them (of his choice), and Joe sees which one Kumail points to. Joe then must pick one of the 64 pennies, and flip it over. Then, Joe leaves the room and Cody enters the room (without talking to each other). Cody looks at the pennies on the chessboard, and tells Kumail which one he had pointed to. Wow!

There's a strategy which Joe and Cody can use to make this happen every time. How?

2 Problems

1. A 2×3 rectangle has vertices at $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

2. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

3. Let
$$T_0 = 2, T_1 = 3, T_2 = 6,$$
and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are
$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$
Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

4. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?