9. Linear Algebra

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1 Warm-up

Calculate the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 5 & 8 & 1 & 0 \\ 8 & 3 & 2 & 9 & 0 & 0 & 3 \\ 7 & 4 & 2 & 4 & 2 & 4 & 1 \\ 8 & 5 & 1 & 6 & 8 & 3 & 2 \end{pmatrix}.$$

2 Classical results

Determinants. The determinant of the product of square matrices is the product of the determinants. The determinant is zero if and only if the rows are linearly dependent.

$\mathbf{SO}(n, \mathbb{Z})$. The set of integer matrices with integral inverses is exactly the same as the set of integer matrices with determinant ± 1.

Diagonalizability. All real symmetric matrices can be expressed in the form PDP^{-1}, where D is a diagonal matrix with all real entries, and P is an orthogonal matrix (satisfying $P^{-1} = P^T$). More generally, all Hermitian matrices (satisfying $A = A^\dagger$, where that is the conjugate transpose) are diagonalizable with P as a unitary matrix (satisfying $P^{-1} = P^\dagger$), as are all skew-Hermitian matrices (satisfying $A^\dagger = -A$).

Eigenvalues, trace, determinant, and rank. The sum of the eigenvalues is equal to the trace (sum of diagonal entries), and the product of the eigenvalues is the determinant. The multiplicity of zero as an eigenvalue is equal to the dimension of the kernel of the matrix.

3 Problems

VTRMC 2012/0. When and where is the VTRMC?

Putnam 1994/A4. Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

Apostol 2.21.4. Show that the only 2×2 matrices A which satisfy $A^2 = A$ are of the form:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix},$$

where x, y, and z satisfy the equation $x(1-x) - yz = 0$.

VTRMC 2000/2. Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (i.e. the sum of the entries on the main diagonal) is an integer.

Trace. Let G be a regular graph (where every vertex has equal degree), where every vertex has at least one edge, and there is no edge from a vertex to itself. Its *adjacency matrix* A is the matrix of zeros and ones where $a_{ij} = 1$ if and only if the i-th and j-th vertices are adjacent. Prove that A has an eigenvalue which is strictly negative.

Putnam 1951/A1. Let A be a real 4×4 skew-symmetric matrix ($A^T = -A$). Prove that $\det A \geq 0$.

VTRMC 1999/3. Let c, M be positive real numbers, and let A be an $n \times n$ matrix with integer entries such that the sum of the absolute values of the entries in each row of A is at most M. If d is a positive real number, let $c_n(d)$ denote the number of nonzero eigenvalues of A which have absolute value less than d. (Some eigenvalues can be complex numbers.) Prove that one can choose $d > 0$ so that $c_n(d) < cn$.

Putnam 1940/B6. The $n \times n$ matrix (m_{ij}) is defined as $m_{ij} = a_i a_j$ for $i \neq j$, and $a_i^2 + k$ for $i = j$. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

Inspired by Putnam. Let A be a square matrix, all of whose entries are strictly positive. Show that if $Ax = x$ has any solutions at all, then it essentially has only one, in the sense that they are all constant multiples of each other.

Putnam 1994/B4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

Putnam 1996/B4. For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

Putnam 1993/B4. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

Putnam 1993/B5. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

4 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.