2011-08-29

~~A~~ A

USAMO 2003/1

5^n | all odd #

$4 + 2^2$ 25: 25, 50, 75

5: 5, 3, 15

125: 125, 250, 375

$4 + 2^3$ 125, $\times 7$, 875

5^4

625: 625 $\times$ 3 = 1875 no.   625, 1250, 1875   up to 2^4 all odd

125 $\to$ 375

625

5^3

· 125×3 is OK.

$125 \times (3+8)$ is OK since $+1000$

$125 \times (3 + 8 \times \text{all odd})$ is OK.

since all odd one will get 5.

all odd: 1, 3, 5, 7, 9 one of these hits 5.

125: 3-digit

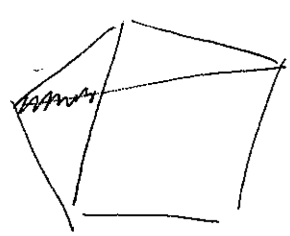
$125 \times (3 + \underline{\quad\quad})$

3 digit. pick 4^{th} digit to be 1, 3, 5, 7, 9 → one choice, ok

$625 \times (\cancel{x} + \underline{2^4 \times D})$

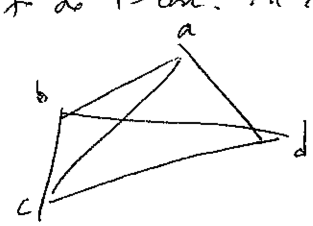
5^4 gives 4-digit. pick 5^{th} digit to be 1, 3, 5, 7, 9

USAMO 2003/2

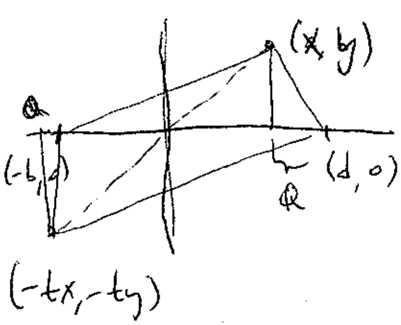


Show all segments from corner endpoint are rational

Suffices to do 4-case. All rational lengths.



$b + t(d-b) = a + u(c-a)$



$(x^2+y^2)(t+1)^2$ is $\mathbb{Q}^2$

$d + b \in \cancel{\mathbb{Q}}\ \mathbb{Z}$

$(d+tx)^2 + (ty)^2 \in \cancel{\mathbb{Q}^2}\ \mathbb{Z}^2$

$(tx-b)^2 + (ty)^2 \in \mathbb{Z}^2.$

$tx \cancel{+} b \in \mathbb{Q}.$

diff: $(d+b)(2tx + d - b) \in \mathbb{Z}.$

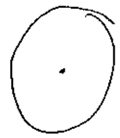
so $2tx + d - b \in \mathbb{Q}$. $\to 2tx \cancel{+} 2b \in \mathbb{Q}$

Rezvan ~~1.1~~ 1.1/4

2018-08-30 (A)

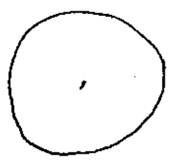
R, G, B 3-space. Complete ~~color~~ distance in color

~~1/2~~ Density version?

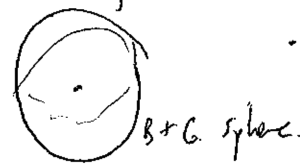


Shells. One shell is ~~actually~~ missing a color

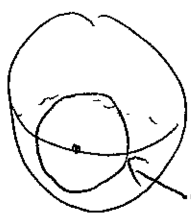
No R in distance 1



R missing distance 1



B+G sphere.

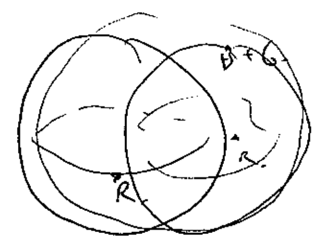


B+G sphere

Say B missing distance $\frac{1}{2}$.

all G ≠ G has continuous range $0 \dots 1$.

If R missing distance d_1
and B missing distance $d_2 < \frac{1}{2} d_1$
then G gets all distances 0 — d_2

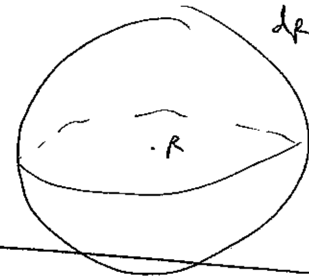


So R has all dists after d_R --- ∞
G d_G --- ∞
B d_B --- ∞.

Why R is missing d_R?

$\boxed{\text{Say } d_R > d_G > d_B}$

d_R



---

2 colors? Even easier.

4 ~~col~~ 2-D space?

Rezvan 1.1/10 Say all $\le n$. Prime factorizations $p_1, p_2, \dots, p_k$.

$k+1$ vectors $\mathbb{Z}^k$. All are ~~≤~~ sum of others.

~~One of ≤ sum~~

$5v_1 - 2v_2 - 10v_3 = 0$. biggest abs value

Also pigeonhole: every ~~...~~ vector exceeds in 1 coord.
So can't have $k+1$

$c_1 v_1 + c_2 v_2 + \dots + c_{k+1} v_{k+1} = 0$

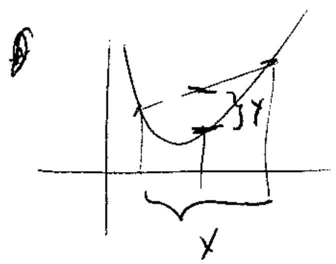
So take ~~smallest coeff~~ biggest c_i. (nonzero)

$c_i v_i = (\ )$ ~~big~~ smaller coeffs $\times v_j$

$v_i = 0.5 v_1 + 0.2 v_2 - 0.1 v_3 \le v_1 + v_2 + \dots + v_{\neq i}$.

2011-08-30 (B)

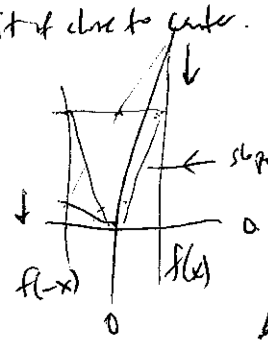
USAMO 2000/1 Very convex



$Y \geq X$.

Sequence: every $\mathbb{Z}$ off by 2. Should get gaps of +2 or more.

Try limit of close to center.



Consider all $\frac{f(y)-f(x)}{y-x}$

Certainly convex, so $\forall y > x$: $\frac{f(y)-f(x)}{y-x}$ is increasing in y

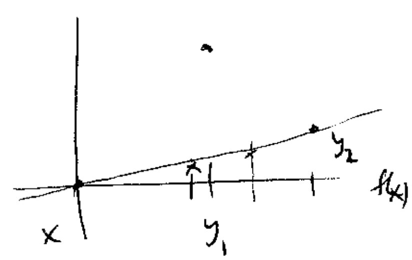
Angle is bounded from below. ⇒ slope has discrete gaps

Take $\lim_n$.

Midpoint -

Convex function: Why is $\frac{f(y)-f(x)}{y-x}$ increasing in y?

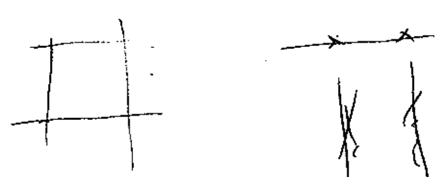
Suppose not:



USAMO 2000/4

999×2. no △.

so conjecture 1999.



(X X) pair adj. Knocks out 99

Note: $2n-1$ gets △

~~Row with~~ 2:

→ ∃ row with only 1 or 0

→ ∃ col with only 1 or 0.

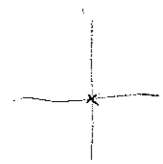
Then take away that row & cols

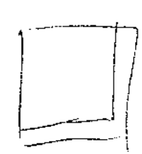
n of these.

Doublecount diagonal

⇒ Maybe split $3n-1$ symbols

⇒ Some has ≤ 2. Use that one



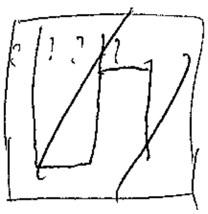


Want row with only 1, column only 1

USAMO 1999/4

2011-08-30

(c)



Say all < 2.

$a_1 + a_2 + \cdots + a_n \geq n$. AVG ≥ 1.

$a_1^2 + \cdots + a_n^2 \geq n^2$ AVG of squares $\geq n$.

RMS $\geq \sqrt{n}$.

$n > 3$.

To maximize $a_1^2 + \cdots + a_n^2$, make as far apart as possible

2, 2, 2, 2, 2, big tiny

2, 2, 2, -2

Move to $a_i - 2$ so all negative

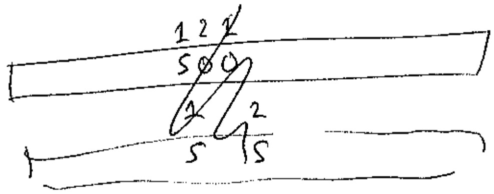
or $b_i = 2 - a_i > 0$.

$b_1 + b_2 + \cdots + b_n = 2n - \sum a_i \leq n$. and all $b_i > 0$.

$(b_1 - 2)^2 + \cdots + (b_n - 2)^2 \geq n^2$

$\sum b_i^2 - 4 \sum b_i + 4 \geq n^2$.

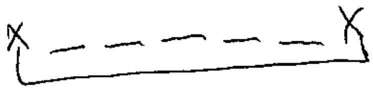
USAMO 1999/5



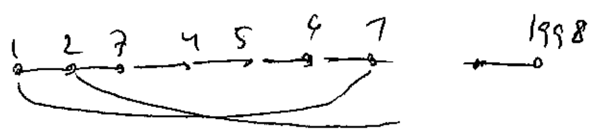
Strategy stealing?

USAMO 1998/1

Need even # of 6.



Graph



etc.

seek perfect matching

Label by left endpoint.

Use parity

Then get rid of

So each +6 type gets both endpts same parity. But if odd # of +6 types, then stuck with diff. parity of residual.

IMO ~~2~~005/2

2011-08-30

~~66~~ 0

Can't have z twice, else for big n, duplicate remainder

So why z appear at all?

Say 0?

IMO 2003/1

Say $a + x = a' + x'$.

Then $x - x' = a' - a$. A is given. Determines A−A forbidden differences $\le 101^2 \approx 10^4$.

Out of 10^6, construct set avoiding these positive differences

$\binom{101}{2}$ positive differences excluded $= 101 \times 50 = 5050$

Pick 1. Block out 5050 ~~[illegible]~~ forward.

Pick another

IMO 2001/4 STS: can't get all $n!$ residues

Say get all $n!$ residues.

Must have distinct k_i's

Full sum of all $S(a)$?

$$\sum_{i=1}^{n} k_i \left[(1+2+\dots+(n)\right](n-1)! \qquad k_i \frac{n(n+1)}{2}(n-1)! = \frac{n+1}{2} n! \sum_{i=1}^{n} k_i$$

~~[illegible]~~ $n=3$.

$a + 2b + 3c$

$a + 3b + 2c$

$2a + b + 3c$

$2a + 3b + c$

$3a + b + 2c$

$3a + 2b + c$

$n!$ is even.

Can't pair

$$x + y = (n+1) \sum_{i=1}^{n} k_i$$

even. so it can be halved!

out of $n!$

mod 6: sum to ~~[illegible]~~ 4

5+5

1+3

2+2

4+0

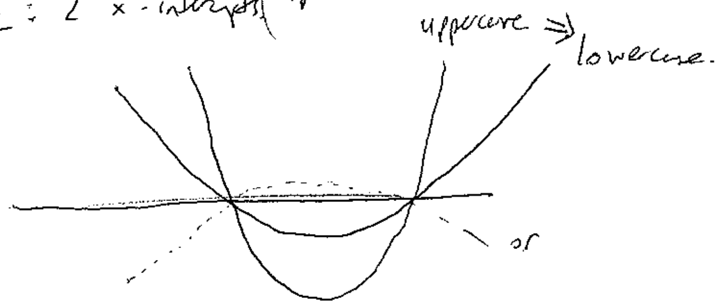
perm ↔ reverse perm

pairing. No identity map.

2003/A4

2011-09-04
(1)

Case 1: 2 x-intercepts (upper case)



So lower is $a(x-r)(x-s) = ax^2 - a(r+s)x + ars$

upper is $A(x-r)(x-s) = Ax^2 - A(r+s)x + Ars$

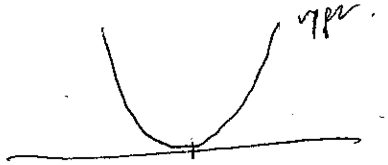
$|A| \geq |a|$.

$b^2 - 4ac = a^2(r+s)^2 - 4a^2 rs.$

$B^2 - 4AC = A^2(r+s)^2 - 4A^2 rs.$

Certainly off by ok.

Case 2 Uppercase has 1 x-intercept:



Lower can't cross through else nonzero derivative
⇒ in e-nbh, it is too big.
So lower has also one intercept,
now $a(x-r)^2$ and $A(x-r)^2$ again.

Case 3. Uppercase has 0 x-intercepts

WLOG both $A, a > 0$.

Actually, then $a \leq A$ else eventually lose

$ax^2 + bx + c$ at $x = -\frac{b}{2a}$.

$$\frac{b^2}{4a} - \frac{b^2}{2a} + \cancel{\frac{c}{2a}} = \cancel{}$$

$$= -\frac{b^2}{4a} + c = -\frac{b^2-4ac}{4a}$$

Now $B^2 - 4AC$ negative.

So if $b^2 - 4ac$ also negative, then:

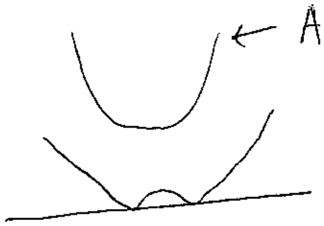
$$\frac{|b^2-4ac|}{4a} \leq \frac{|B^2-4AC|}{4A}$$ since min lowercase $\leq$ min uppercase

$$\Rightarrow |b^2-4ac| \leq \frac{a}{A}|B^2-4AC| \leq |B^2-4AC|.$$ ✓

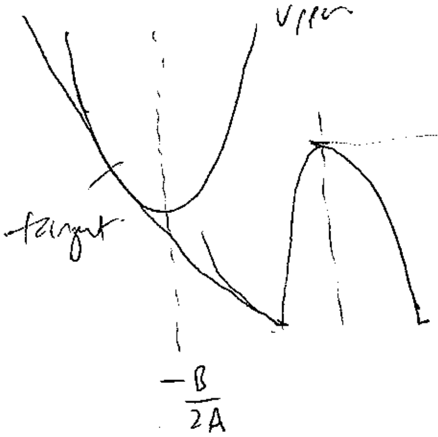
Case 4. Upper: $\leftarrow A\left(x-\frac{B}{2A}\right)^2+\frac{B^2-4AC}{4A}$

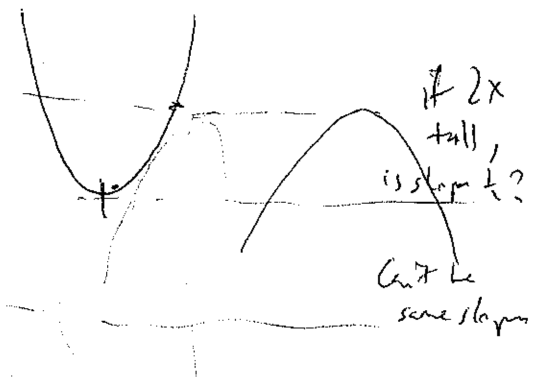
2011-09-04

2



Lower.





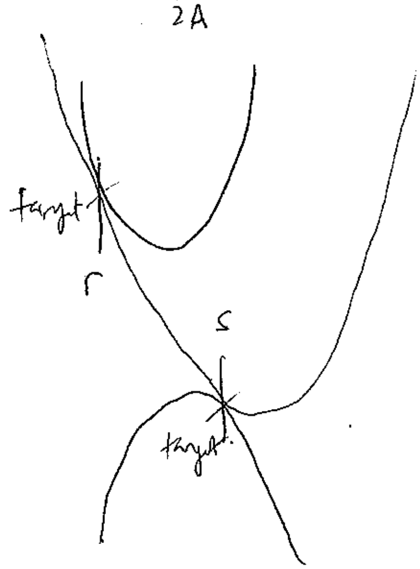
Brute force. Given Ax^2+Bx+C.

At $x=r$, derivative is $2Ar+B$.

$x=s$, $2As+B$

So
$$\begin{cases} 2Ar+B = 2ar+b. \\ -2As-B = 2as+b. \end{cases}$$

and.
$$\begin{cases} Ar^2+Br+C = ar^2+br+c \\ -As^2-Bs-C = as^2+bs+c. \end{cases}$$



$\cancel{2A(r+s) = 2a(r-s)}$

$2A(r+s)+2B = 2a(r-s)$

$$a = \frac{A(r+s)+B}{r-s}$$

$$b = 2Ar+B - 2\frac{A(r+s)+B}{r-s}r.$$

$$c = Ar^2+Br+C - \frac{A(r+s)+B}{\ }$$

$\cancel{2A(r-s) = 2a(r-s)}$

$\cancel{A \neq a}$

So can't tangent

mess. want: $b^2-4ac \le -B^2+4AC$

$(A-a)x^2+(B-b)x+(C-c) \geq 0.$

$(A+a)x^2+(B+b)x+(C+c) \geq 0.$

$(B-b)^2-4(A-a)(C-c) \le 0.$

$(B+b)^2-4(A+a)(C+c) \le 0.$

$\oplus$

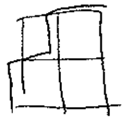
$2B^2+2b^2-8AC-8ac \le 0$

$B^2-4AC \le -b^2+4ac.$

$\square$.

2003/A5 $\quad D_1 = 1$ $\qquad$ 2011-09-04 ③

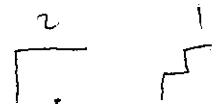
Dyck 2-paths No even return



Dyck 1-path:  $= 1 = D_1$

Dyck 3 paths no even return

Dyck 2-paths



$D_2 = 2$.

D_3:

5.

Dyck 4. no even return

$X_n = A_1 X_{n-1} + A_2 X_{n-2} + \dots A_n X_0$

$= A_1 C_{n-2} + A_2 C_{n-2} + \dots + A_{n-1} C_0 + A_n$

C_0

basically, we do

X_n

$\cancel{X_n = X_{n-1} + X_{n-3} + X_{n-5} + \dots}$

no even returns.

It should be C_{n-1}. //

with even returns should be $C_n - C_{n-1}$.

$\binom{2n}{n} - \binom{2n}{n-1} - \binom{2n-2}{n-1} + \binom{2n-2}{n-2}$

D_n = catalan (n)!

$\cancel{= D_{n-1} + D_{n-2} + D_{n-3} + \dots + D_1 + D_0}$

Z_n = length n, only one return

$= D_{n-1}$.

$Z_1 D_{n-1} + Z_2 D_{n-2} + \dots + Z_n D_0$.

$= D_0 D_{n-1} + D_1 D_{n-2} + \dots + D_{n-1} D_0$.

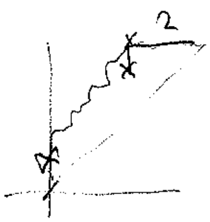
Y_n: no even return, actually, just only odd return at end.

$=$: Catalan

A_{n-1}: with even return at end

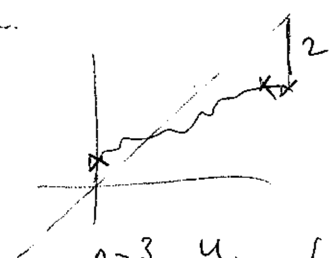
$\sum D_n z^n = z \sum_n \sum_i (D_i z^i)(D_{n-i} z^{n-i})$

$= \dots$

A_n: 

Reflection principle.

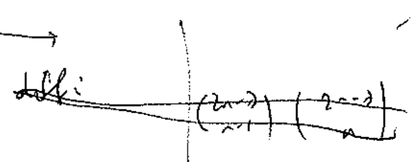
How many hit line? Reflect after first hit.



$\begin{array}{l} n-2 \ U. \\ n-2 \ R. \end{array} \to \binom{2n-4}{n-2} \longrightarrow$

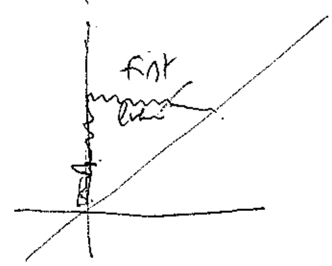
$\begin{array}{l} n-3 \ U. \\ n-1 \ R. \end{array} \binom{2n-4}{n-1}$ smaller.

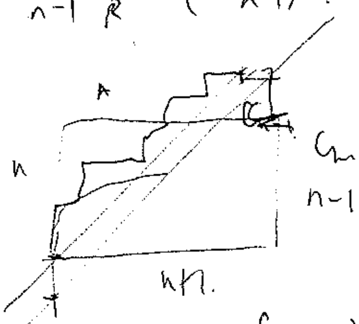
$\begin{array}{l} n-2 \ U \\ n-4 \ R \end{array} \to \binom{2n-6}{n-2}$



$\begin{array}{l} n-5 \ U \\ n-1 \ R \end{array} \binom{2n-6}{n-1}$.

$\binom{2n-4}{n-2} - \binom{2n-4}{n-1} + \binom{2n-6}{n-2} - \binom{2n-6}{n-1}$





$\binom{2n}{n} - \binom{2n}{n-1}$

$= \binom{2n}{n} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!\,n!} \frac{n}{n+1}$ so $\binom{2n}{n}\left[1 - \frac{n}{n+1}\right]$

$= \frac{1}{n+1}\binom{2n}{n}$

2011-09-04

(4)

2003/A5

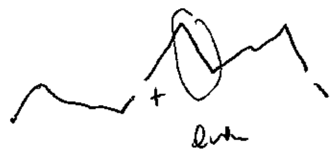
Bijection: Dyck n-1:

 $\longmapsto$ + ∧ at end

if no even return

f

if ∃ even return

 $\longmapsto$ 

even even

even

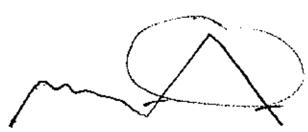
~~injection~~

g: look on return to end. Then pull it down

$\xrightarrow{f}\xrightarrow{g}$ gets back to original.

$\xrightarrow{g}\xrightarrow{f}$ ~~?~~.

but why it hits everyone with no even return?

no even return

pull down should get even return in there.

2003/A6

A 0 3 5 even #1's 11001$0_2$.

B 1 2 4 odd #1's. bijection

even #1's: 1 0 1 0 1 1 $_2$?

+ 0 0 0 0 1 1 $_2$ forced.

well-defined. it. flip them.

why unique? 100011

if was 1100011

?0 00 11

~~0+1~~ 0+3

~~0+2~~ 1+2.

1+1.

0+4 0+5

1+3 1+4

odd #1's 2+2 2+3

2011-09-04
(5)

2003/A6

why bijection?

even # 1's vs odd # 1's.

To get sum 5: 100_2. 1's. 000_p + 011

$\boxed{000} \longleftrightarrow 001$

$011 \longleftrightarrow 010.$

why not get 001 any other way? Must come from even # of 1's.

how it got there $\xleftarrow{\text{invert}}$ 001 / $\boxed{010}$ forced (sum 5) + 011 / 001 $\longrightarrow$ 001 / 011. sum is not 5.

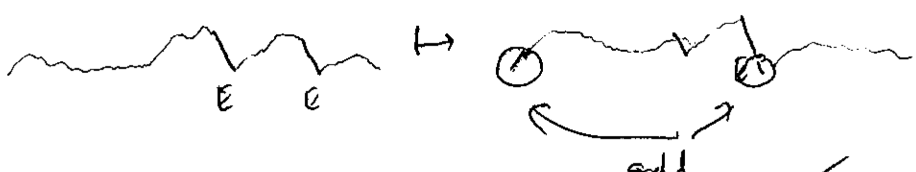
2 well-defined maps. Clearly ~~inverse~~ couple to get id.

2003/A5

C_{n-1}
no even return

Not symmetric!!

YES:



Trust: odd return gives even return

MC State 91

2001-09-04

6

(1) 32, 33, 34

(2) $\frac{3}{4}\times\frac{2}{3}=$ 48

(3) 18

36

$2\times\pi\left(\frac{18}{\pi}\right)^2=\frac{324\times 2}{\pi}$

$\frac{648}{\pi}$

(4) $\left(\frac{2x}{\sqrt{3}}\right)^2 6=\frac{4x^2}{3}\times 6=8x^2$

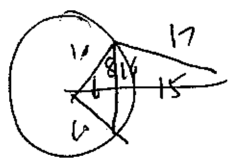
(5) $1+3\times 9=28$

(6) $10\times 12\rightarrow 7\times 9$

120 63

$\frac{57}{120}=\frac{19}{40}$

(7)

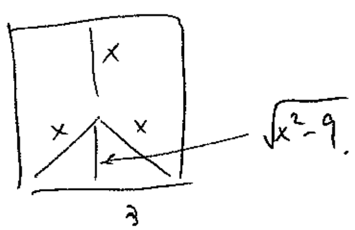


21

(8) $\frac{4}{5}\times 150\times\frac{3}{2}=180$

(9) $\frac{3x+6}{3}=7$ $x+2=7$ 5

(10)



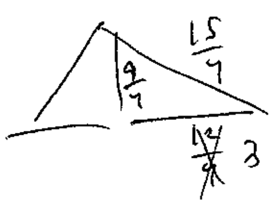
$x+\sqrt{x^2-9}=6$

$\sqrt{x^2-9}=6-x$

$\cancel{x^2}-9=36-12x+\cancel{x^2}$

$12x=45$

$x=\frac{45}{12}=\frac{15}{4}$



$\frac{27}{4}$

2011-09-04
①

1994/B1 all $\mathbb{Z}^+$ within 250 of exactly 15 perfect squares

0 $\cancel{\boxed{1,2,3,4,5}}$ ... 255 + $\cancel{\boxed{82 ... 100}}$

sq.

Only lose 1 when get to 251.

+ 250

= $\boxed{332 ... 350}$

General case:

L x L+250

x−250 x+250

15 squares?

Squares: $z^2, \ldots, (z+14)^2$

Need $(z+14)^2 - z^2 \le 500 \longrightarrow (2z+14)\,14 \le 500$

$2z+14 \le \frac{250}{7}$

$2z \le \frac{152}{7}$

$z \le \frac{76}{7} = 10\frac{6}{7}$

$$\begin{array}{r} 35 \\ 14\,\overline{)500} \\ \underline{42} \\ 80 \\ \underline{70} \\ 10 \end{array}$$

$7 \times 14 = 98$

but $(z+15)^2 - (z-1)^2 \ge 501$

$(2z+14)\,16 \ge 501$

$2z + 14 > \frac{500}{16} = \frac{125}{4}$

$$\begin{array}{r} 125 \\ -56 \\ \hline 69 \end{array}$$

$2z > \frac{69}{4}$

$z > \frac{69}{8} = 8\frac{5}{8}$

so must have $z = 10$ exactly.

Only way is to hit

9^2 (10^2 ... 24^2) 25^2

81 | 100 576 625

L=82 ... L=100.

or

8^2 (9^2 23^2) 24^2

64 | 81 529 576

65 75

+ 250

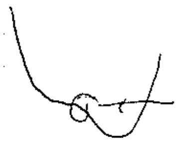
= $\boxed{315 ... 325}$

1994/B2 $\quad x^4+9x^3+cx^2+9x+\cancel{4}$ hit 4 points

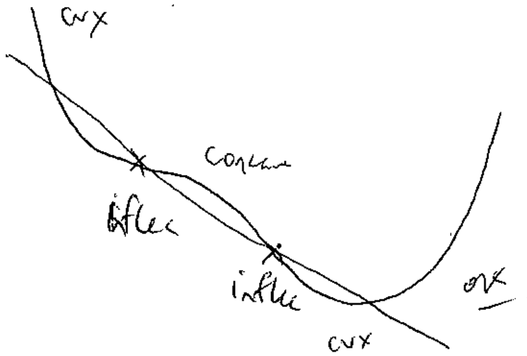
2011-09-04

②

Symmetric? No.



any bump at all will do.



no inflec.

inflec.

Inflection: 2nd derivative is 0.

Der 1: $\quad 4x^3+27x^2+2cx+9.$

Der 2: $\quad 12x^2+54x+2c.$

$$6\cdot\left[2x^2+9x+\frac{c}{3}\right]. = 12\left[x^2+\frac{9}{2}x+\frac{c}{6}\right].$$

~~Discrim~~

$$=12\left[x^2+\frac{9}{2}x+\left(\frac{9}{4}\right)^2+\frac{c}{6}-\frac{81}{16}\right].$$

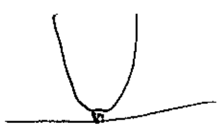
So for $\frac{c}{6}-\frac{81}{16}<0$, definitely 2 pts of inflection

$$\frac{c}{6}<\frac{81}{16}$$

$$c<\frac{81\times 3}{8}=\frac{243}{8}.$$

$$c<\frac{243}{8}$$

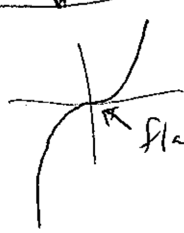
What if Der 2 is



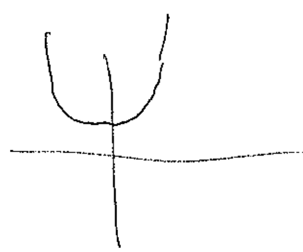
cvx throughout.

∴ only 2 hits

Then Der 1 is



Der 0 is



2011-09-06

(1)

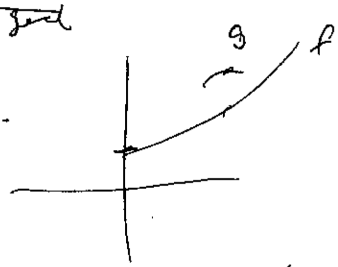
1994/B3

$f'(x) > f(x)$, $\forall x$. Eventually beats e^{kx}.

Same initial conditions,

~~Say negative for good~~

Positive f.



$f' > 0 \Rightarrow f' > 0 \Rightarrow f$ inc $\Rightarrow f'$ incr $\Rightarrow$ cvx

$$\int_0^t \frac{f'(x)}{f(x)}\,dx > \int_0^t 1\,dx$$

g same init cond but $g'(x) = g(x)$

$f'(x) - g'(x) \geq$ ~~0 always.~~

$\ln f(t) - \ln f(0) > t$.

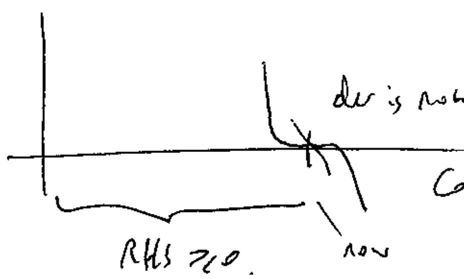
$f(t) > f(0)\, e^{t}$. ✓.

$f'(x) - g'(x) > f(x) - g(x)$

At $x=0$, RHS $=0$.

RHS is continuous so at some point it's <0

Inf of time it is <0.



So $f(x) > g(x) = Ce^{x}$. So true for all $k<1$ for sure

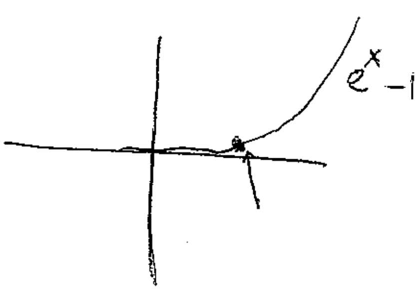
How about $k=1$?

$f(x) = e^{x} +$ thing

$f(x) = e^{x} + e^{-x}$

$f'(x) = e^{x} - e^{-x}$. Not bigger

$f(x) = e^{1.1x}$ shows $k>1$ won't work



$f(x) = e^{x} - 1$.

$f'(x) = e^{x}$. true. Never beats e^{x}.

So $k<1$

$e^{x} - \frac{1}{x}$? Der is $e^{x} + \frac{1}{x^2}$.

$f = e^{x} - e^{-x^2}$? $f' = e^{x} + 2x e^{-x^2}$

$f = e^{x} - e^{-x^3}$ $f' = e^{x} - e^{-x^3}(-3x^2)$

$= e^{x} + 3x^2 e^{-x^3}$

Conclusion

Won't tenure at Princeton/Harvard/MIT/Stanford/...

Technology lets us have opportunity for departmental expansion through junior candidates.

Specifically beat our targeted level of Cornell/Yale +

Junior only, but network lets us vet them before search.

Build home base for network, start by attracting builders.

Give them ~~room~~ space to innovate, and collaborative environment

Proof of concept will spread to other departments as well -- Hertz network and more.

OK ~~In the end, will add new star to university~~

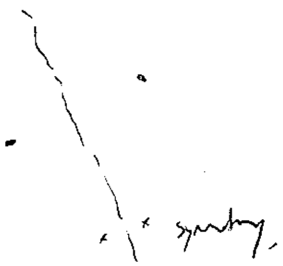
Inspire Innovation Process in working that ~~could~~ could lead to new growth strategy for

Many depts in the university. junior vs senior hires.

IMO 1999/1

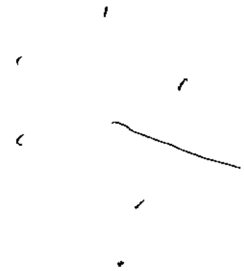
2011-09-06

(1)



Focus on:

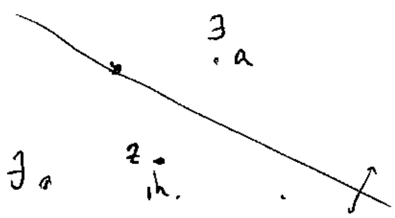
Convex hull.



If ∃ point inside cvx hull...

before & after reflection

Use fact that cvx hull is well-defined in both objects and must therefore coincide

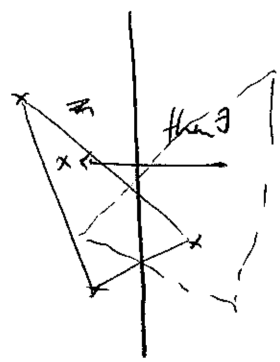


i.e. Convex hull maps to itself in the reflection.

But now $a \leftrightarrow z$, so fail, since a in cvx hull but z not.

CHECK. In any reflection, is it true that pts in cvx hull must map to each other?

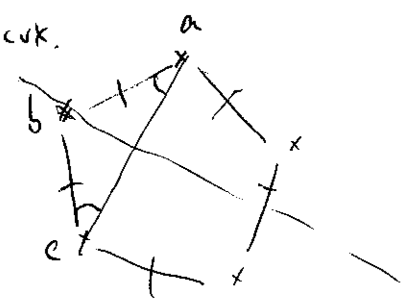
say:



so wasn't in cvx hull originally either

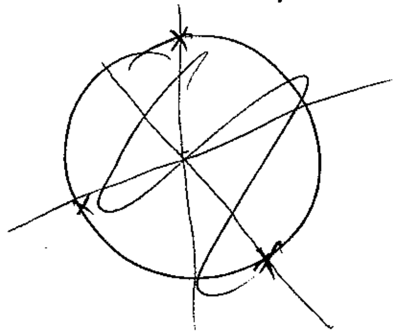
Hence we can't afford to swap cvx hull point with internal point.

⇒ set is cvx.

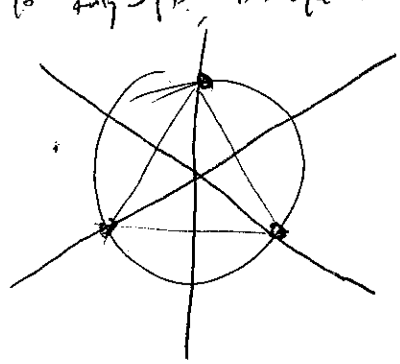


Apply to pts 2 apart. forces middle (b) to be on ⊥ bis. ⇒ ab = bc

Make Circumcircle of 3 pts.



If only 3 pts, it's equilateral



Got equilateral!

If n ~~places~~ odd, then pts every 2 too.

⇒ congruent Δs

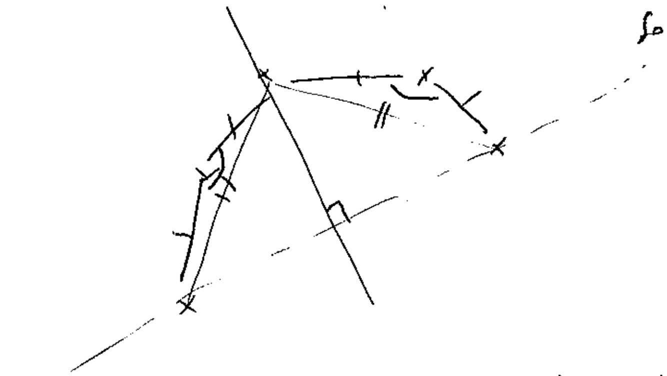
⇒ equiangular

⇒ regular

by go around by 2s.

2011-09-06
(2)

So if $n \neq 4$, can do:

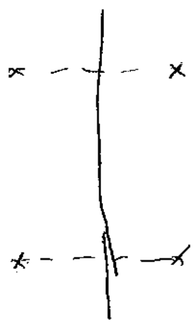


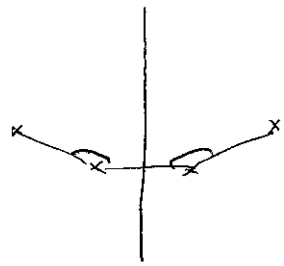
So alternating angles are equal.
Solves all odd n.

Oh, get equal angles directly from bisecting consecutive edge. ✓

Remains: $n=4$, all even n.

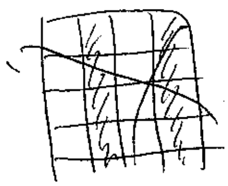
$n=4$:





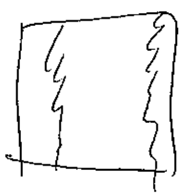
So regular

IMO 1999/3

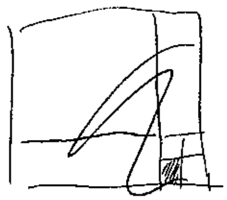


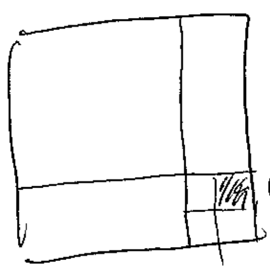
OK

Induction?



Cracks, get $\frac{n^2}{2}$ or





WLOG.

IMO 1999/4 $n \le 2p$. $(p-1)^n+1$ div by n^{p-1} $n=1$

$n^{p-1} \equiv 1 \pmod p$ $(pk+1) \mid (p-1)^n+1$

$p=2$: 2 div by n. $\Rightarrow (n,p) = (1,2), (2,2)$ $\cancel{n \ge p \quad (-1)^p+1}$

$p=3$: 2^n+1 div by n^2. Need $(p-1, n) = 1$. $n=3$ works

$(p-1)^p+1 = p^p - p\,p^{p-1} + \binom{p}{2} p^{p-2} - \cdots -1+1$.

$p=5$: 4^n+1 div by n^4. Need n odd. ~~Already won't try $n \ge 5$~~

$3^4 > 4^3$. So won't try anything $n \le p-1$ in general.

$2^4 = 4^2$. Check range $n = p \ldots 2p$.

$2^5 > 5^2$

IMO 1997/3 2011-09-06
(3)

$|\text{Sum}| = 1$. Each $|x_i| \le \frac{n+1}{2}$ Induction?

$n=1$: trivial obvious.

Get $|y_1 + 2y_2 + \dots + ny_n| \le \frac{n+1}{2}$.

Obviously sort in reverse order. No, can have ⊖.

Can get $|y_1 + \dots + (n-1)y_{n-1}| \le \frac{n-1}{2}$. ~~What tends by ±~~.

Apply to previous, need to scale x_i's by $\times \frac{n-1}{n+1}$.

So can get, then reverse scale $|y_1 + \dots + (n-1)y_{n-1}| \le \frac{n-1}{2} \times \frac{n+1}{n-1} = \frac{n+1}{2}$.

Or get $|2y_1 + \dots + (n)y_{n-1}| \le \frac{n-1}{2} + \frac{n-1}{n+1}$

reverse scale ... $\le \frac{n+1}{2} + 1$.

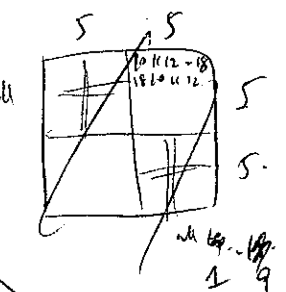
But now we push it by the biggest x_i.

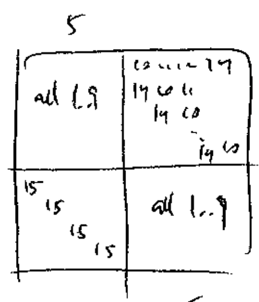
~~Yet, so we pull out biggest x_i. Sort by reverse absolute value~~

WLOG say sum is positive $= +1$.

Pull out biggest x_1. Now sum could be like $1 - \frac{n+1}{2}$, really high.

IMO 1997/4 Silver construction.



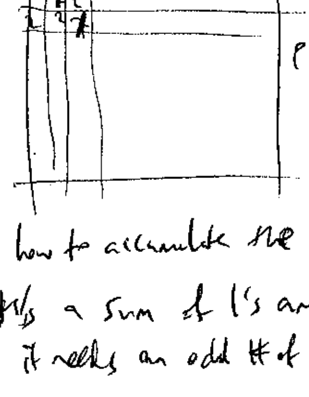


10 11 12 13 14
15 16 17 18 19

So can $\times 2$

Hadamard Matrix construction.

→ no 1997 prime.



Each item must appear exactly p times.
There are $2p-1$ items
So total appears is $p(2p-1)$

But how to accumulate the p times? per $2p-1$ guy?
It's a sum of 1's and 2's.
So it needs an odd # of 1's. Yet there are only p on diagonal to go around ⨳.

2011-09-06
(4)

IMO 1993/5

$f(1)=2$. $\quad f: \mathbb{Z}^+ \to \mathbb{Z}^+$.

$f(f(n)) = f(n) + n$ and $f(n) < f(n+1)$.

$f(2) = \cancel{\ } 2+\cancel{1} = 3$

$f(3) = 3+2 = \cancel{\ } 5$.

$f(5) = 5+3 = 8$.

$f(8) = 8+5 = 13$. Fibonacci. $\longrightarrow$ so $f(k) \approx 1.618k$ $\cancel{f(k) = \{1.618k\}}$.

$f(4) = 6$?

$\Rightarrow f(6) = 6+4 = 10$. $\quad a_n = a_{n-1} + a_{n-2}$ with diff. initial conditions

$f(10) = 10+6 = 16$. $\quad f(k) = \{1.618k\}$? Round to nearest.

$f(16) = 26$ etc.

$f(\text{nearest to } 1.618n) = \text{nearest to } 1.618n + n$.

Say $1.618n = z+r$. $\quad 1.618z$ nearest vs $z+n$

nearest to $1.618(1.618n - r)$ vs. $(1.618n - r) + n$

$2.\cancel{618}n - 1.618r$. $\quad \cancel{2.618}n - r$.

Say $r < \frac{1}{2}$ positive

It's like

rounded down

$m.41 - 1.618 \times 0.41$, round. $\quad$ So $2.618n$ is like $m.41$

Note that even $0.50 - 1.618 \times 0.5$

nets -0.3 ish, so rounds up.

Say it was like $m.60$

say negative $r > -\frac{1}{2}$. $\quad$ Same $\quad r = -0.40$

2011-09-06

5

IMO 92/1

$$a+b+c-ab-ac-bc+(abc-1) \mid (abc-1)$$

$$\Updownarrow$$

$$a+b+c-ab-ac-bc+(abc-1) \mid a+b+c-ab-ac-bc$$

IMO 92/5

$\in S_y$

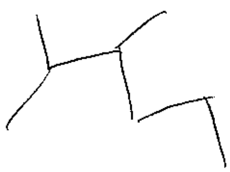
$$~~(x,y,z) \to (x,y,z,x,y,~~ \quad (x, \boxed{x, y,}\boxed{y, z,} z)$$

$\in S_z \quad \in S_x$

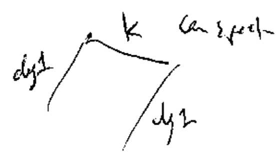
$$\left.\begin{matrix}(x_1,y_1,z_1)\\(x_2,y_2,z_2)\end{matrix}\right] S^2 \longrightarrow (x_1,x_2,y_1,y_2,z_1,z_2)$$

IMO 91/4

GCD to be 1 trees are trouble Or get complete at each?



If remove still got same # vtx,



IMO 91/6

$$|x_i - x_j| \ge \frac{1}{|i-j|^4}$$

GA ~~EG~~ 22

$m_2 = 1^2+2^2 = 5.$ $\quad 5^1$

$m_3 = 5^2+10^2 = 3^2+4^2+10^2 = 125$ $\quad 5^3$

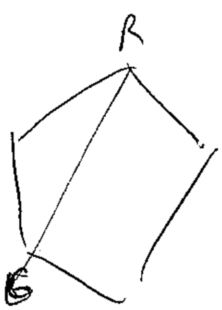
$m_4 = \cancel{\ldots} = 25\, m_3.$ $\quad 5^5$

$5^5 = 5\times 5^4 = 1^2\times 5^4 + 2^2\times 5^4$ $\quad 5^7$

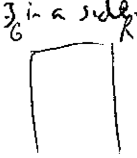
$\quad = 1^2\times 5^2 \times (3^2+4^2) + 2^2\times 5^4$ $\quad 5^9$

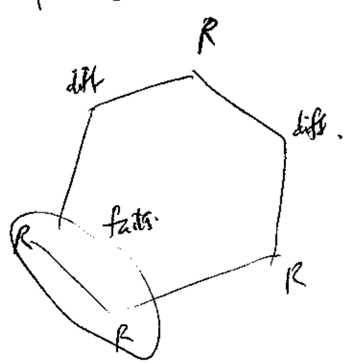
$\quad = 3^2\times(3^2+4^2) + 5^2\times 4^2 + 2^2\times 5^4.$

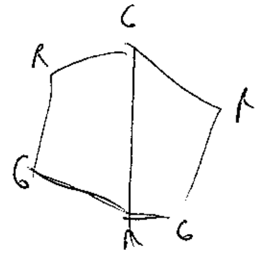
GA 28



STS ∃ diff colors at all.
will ≥3 in a side.





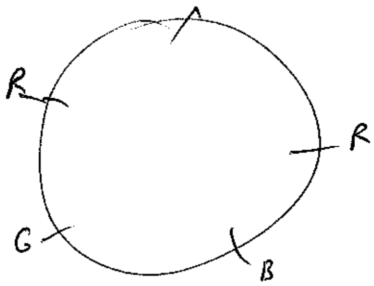


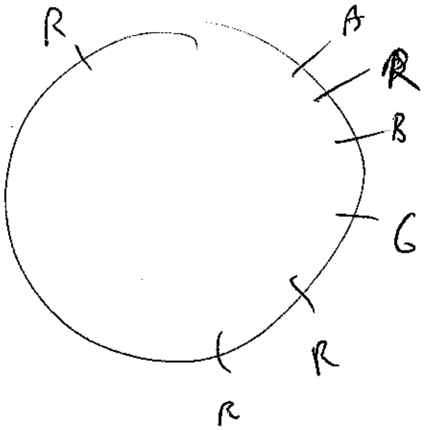
2011-09-06

10

GA 28

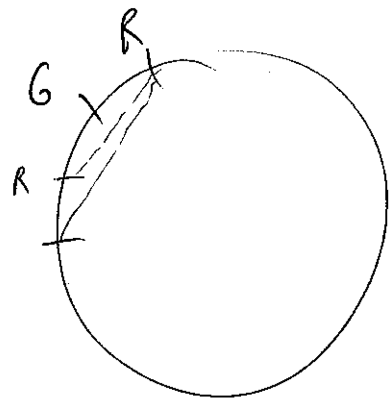
n=4. Use opposite diagonal. If always same color then ≤ 2 total colors





only here B here

only here G here



Dollar Bank 412-913-5903

Randi Weinstein.

\$500 off closing if get Free Checking Acct.

3400 already includes 500.

30 days ~~pts~~ prepaid interest.

2700.

3.25% interest rate

2700.

GA 30

INDUCTION

2011-09-06

(A)

Hadamard: $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$.

Then take $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$.

USAMO 2003/1

$n=1$: true.

Say true for n. Try for $n+1$.

$10^n = 1\underbrace{00000}_{n}$ ← $(n+1)$st digit.

~~Need~~

So, $5^n k$ is n-digit # with all digits odd.

Take $5^n k (5t)$.

or $5^n k\left[1 + 2^n t\right] = t\boxed{\text{old \#}}$

So we could try any of $t = 1, 3, 5, 7, 9$ to get all odd

But one of these will make $1 + 2^n t$ to be div. by 5.

Since 2^n is not div. by the prime 5, ~~and we take~~

So:

$$\left.\begin{matrix} 1 + 2^n \\ 1 + 2^n + 2\times 2^n \\ 1 + 2^n + 4\times 2^n \\ 1 + 2^n + 6\times 2^n \\ 1 + 2^n + 8\times 2^n \end{matrix}\right]$$

K residues are moving up by $+K$, and $K \not\equiv 0$, 5 prime

2011-09-06

2

GA 22 Suppose that can be done for n.

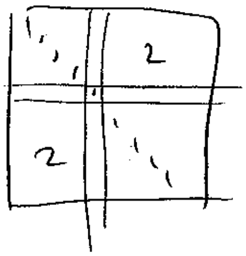
For $n+1$, just take (old #) $\times 5^2$.

Then its last n expression has all squares mult of 5.

And we can peel off the 1st one using $(5x)^2 = (3x)^2 + (4x)^2$.

USAMO 1997/4

(a). 1997 is odd



Total # of times we cover each square is $2x$, except for the diagonal.

But each of $2n-1$ symbols is seen exactly n times.

So when we saw it, each off-diagonal gave $+2$.

each diagonal gave ~~+2~~ $+1$

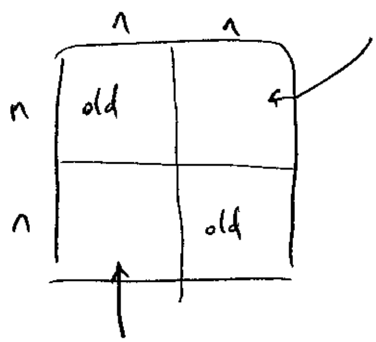
n odd $\Rightarrow$ need each of $2n-1$ symbols to come at least once on the diagonal $\times$

(b). $n=2$:

1	2
3	1

Now build powers of 2:

Say can do n.



$2n \quad (2n+1) \cdots \quad (3n-1)$

$(3n-1) \quad 2n \quad (2n+1)$

$2n$

$2n$

$2n$

$3n \quad (3n+1) \cdots (4n-1)$

$3n$

$(4n-1) \quad 3n \quad (3n+1)$

$3n$

ZUMING 97

Say $\sqrt{a+\sqrt{2a+\sqrt{3a}}} < \sqrt{a}+1$.

Then $\sqrt{2a+\sqrt{4a+\sqrt{6a}}} < \sqrt{2a}+1$.

So $\sqrt{a+\sqrt{2a+\sqrt{3a+\sqrt{4a}}}} < \sqrt{a+\sqrt{2a}+1}$

$< \sqrt{a+2\sqrt{a}+1}$

$= \sqrt{a}+1$

VTRMC 2005/2

2011-09-13

1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	2
3	2	1	4	11	10	9	8	7	6	5	20	19	18	17	16	15	14	13	12
~~15~~	~~14~~	~~13~~	~~12~~	11	10	9			6	~~5~~	20	19	18	17					
~~7~~	~~6~~	~~5~~	4	3	2	1			~~5~~		4	3	2	1					
3	2	1	1																
1																			

only left

only occurrences

only 20

only occurrences

VTRMC 2005/4

$(0,1,2)$ start. Slope is $+1, +2, +2$

First hit $z=7$. at $\left(\frac{5}{2}, 6, 7\right)$ Now flip, so target is $(0,1,14-2)=(0,1,12)$

Next hit $y=7$. At: $(3, 7, 8)$ target is $(0,13,12)$

Next hit $z=14$ At: ~~(4, 29, 30)~~ target is $(0,13,16)$

$(6,13,14)$

target is $(0,15,16)$

Next $y=14$: At $(6.5, 14, 15)$ $(14,15,16)$

Next $x=7$: $(7,15,16)$ $(14,15,26)$

Next $z=21$: $(9.5, 20, 21)$ $(14,27,26)$

$y=21$ $(10, 21, 22)$ $(14,27,30)$

$z=28$ $(13, 27, 28)$ $(14,29,30)$

$y=28$ $(13.5, 28, 29)$

$x=14$ $(14, 29, 30)$ hit.

0 +2

T

14 -2 +2

~~21~~

28 -2 +2

35

42 -2

So it went from $(0,1,2)$ .... $(14, 29, 30)$

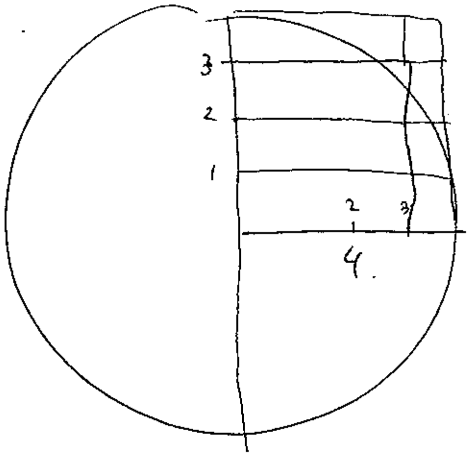
distance: $\langle 14, 28, 28\rangle$

i.e. $14\times\langle 1,2,2\rangle = 14\times\sqrt{2^2+2^2+1^2} = 14\times 3$

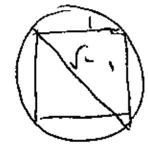
$= 42$

2011-09-13 ②

GA 44



60 circles of diameter $\sqrt{2}$.



$3^2 = 9$. $\sqrt{16-9} = \sqrt{7} \not< 3$..

Got $15 \times 4 = 60$.

GA 46. $V - E + F = 2$.

2 faces with same # of edges?

$\sum \text{degrees} = 2E = \sum_{\text{faces}} \text{perimeter}$

Every graph has 2 vtxs of equal degree

$\downarrow$

Dual.

if degrees are $0, 1, 2, \ldots, n-1$.

Impossible since 0 & $n-1$ ✱.

Hamming code? Hats. ~~Hamper~~ 00011100111

If you see codeword then guess NOT it. Win if not codeword

Then only fail on codewords.

$\frac{d}{d+1}$ success.

Can we get 2^4? All nonzero vectors

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 0.$$

7 coords.

rank 3

Minimal codeword?

No identical, so can't zero out with H-weight 2

So got 2^4 codewords out of 2^7.

This is $\frac{1}{8} = \frac{1}{7+1}$.

$\boxed{\text{all 15 nonzero 4-vectors}}$

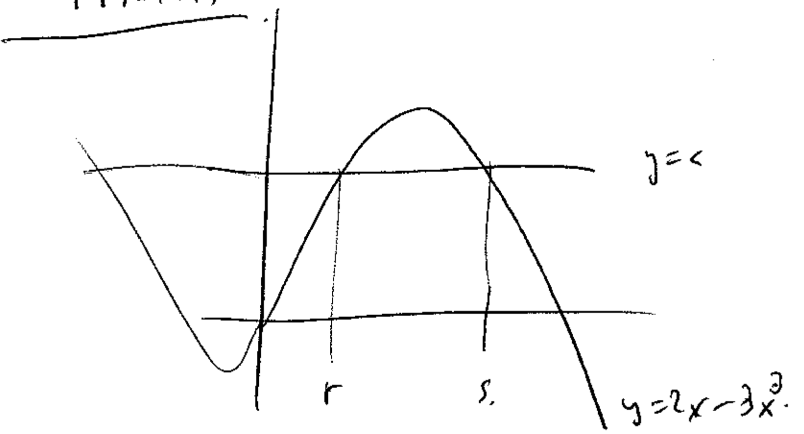
rank = 4

get $2^{15-4} = 2^{11}$.

out of 2^{15}. $\longrightarrow \frac{1}{2^4} = \frac{1}{16}$ of them

2011-09-14

1993/A1



$$\int_0^r c-2x+3x^3\,dx = \int_r^s 2x-3x^3-c.$$

$$\cancel{cr}-\cancel{r^2}+\cancel{\tfrac{3}{4}r^4} = s^2-\cancel{r^2}-\tfrac{3}{4}(s^4-\cancel{r^4}) - c(s-\cancel{r})$$

~~Zero~~

$$0 = s^2 - \frac{3}{4}s^4 - cs.$$

$$0 = s - \frac{3}{4}s^3 - c.$$

But plug in $\underline{s}$: $c = 2s - 3s^3 \longleftrightarrow c = s - \frac{3}{4}s^3.$

$$s = \left(3-\frac{3}{4}\right)s^3 = \frac{9}{4}s^3$$

$$\frac{4}{9} = s^2$$

$$\frac{2}{3} = s$$

$$c = 2\times\frac{2}{3} - 3\times\left(\frac{8}{27}\right)$$

$$= \frac{4}{3} - \frac{8}{9}$$

$$= \frac{4}{9}$$

1993/A2 $x_n^2 - x_{n-1}x_{n+1} = 1.$ for all $\underline{n}$. Need $x_{n+1} = ax_n - x_{n-1}.$

$$x_n^2 z^{2n} - x_{n-1}z^{n-1}x_{n+1}z^{n+1} = z^{2n}$$

$$x_1^2 - x_0x_2 = 1.$$

$$x_2^2 - x_1x_3 = 1.$$

$$x_{n+1} = \frac{x_n^2-1}{x_{n-1}}$$

$$x_{n+2} = \frac{x_{n+1}^2-1}{x_n}$$

$$=$$

$x_0 = 2$
$x_1 = 2$
$x_2 = \frac{3}{2}$
$x_3 = \frac{5}{8}$
$x_4 =$

2011-09-14

(B)

$$x_{n+1} = \frac{x_n^2 - 1}{x_{n-1}} \qquad x_2 = \frac{x_1^2 - 1}{x_0} \overset{!}{=} a x_1 - x_0$$

~~Say x_n~~

$$\frac{x_1^2 + x_0^2 - 1}{x_0} = a x_1$$

$$\frac{x_1^2 + x_0^2 - 1}{x_0 x_1} = a$$

$$x_{n+2} = \frac{x_{n+1}^2 - 1}{x_n} = \frac{(a x_n - x_{n-1})^2 - 1}{x_n} = a^2 x_n - 2a x_{n-1} + \frac{x_{n-1}^2 - 1}{x_n}$$

$$\frac{x_{n+1}^2 - 1}{x_n} \overset{!}{=} a x_{n+1} - x_n$$

$$x_{n+1}^2 - 1 \overset{!}{=} a x_{n+1} x_n + x_n^2 \qquad \text{So need } \frac{x_{n+1}^2 + x_n^2 - 1}{x_{n+1} x_n} \text{ constant}$$

$$\frac{x_{n+1}^2 + x_n^2 - 1}{x_{n+1} x_n} \overset{!}{=} a.$$

$$\frac{x_{n+1}^2 + x_n^2 - 1}{x_{n+1} \cancel{x_n}} \overset{!}{=} \frac{x_n^2 + x_{n-1}^2 - 1}{\cancel{x_n} x_{n-1}}$$

$$\Uparrow$$

$$x_{n+1}^2 x_{n-1} + x_n^2 x_{n-1} - x_{n-1} \overset{!}{=} x_n^2 x_{n+1} + x_{n-1}^2 x_{n+1} - x_{n+1}$$

$$\Uparrow$$

$$x_{n+1}(x_n^2 - 1) + x_n^2 x_{n-1} - x_{n-1} \overset{!}{=} x_n^2 x_{n+1} + x_{n-1}(x_n^2 - 1) - x_{n+1}$$

True.

1993/A3 $\quad c(n,1)$: $\to 1$ so only 1 function

level sets are closed under intersection.

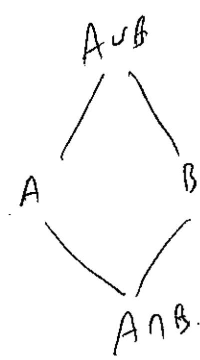
Let $z = f(\emptyset)$. Now $f(A)$ or $f(A^c)$ is z.

$f(\emptyset) = \min(f(\emptyset), f(B))$ so ~~B~~ z is minimal value

$f(B) = \min(f(\text{full}), f(B))$ so $f(\text{full})$ is maximal value

$f(C)$ = maximal value out of all $S \subseteq C$. Since $f(S) = f(S \cap C) = \min(f(S), f(C))$

$f(C)$ = minimal value out of all $C \supseteq S$



1993/A3

2011-09-14

Take 1-element sets

$\{1\}, \{2\}, \{3\}, \ldots, \{n\}$. If $\exists$ 2 not at $\mathbb{Z}$-level, then isect them

$\Rightarrow$ only one can be above $\mathbb{Z}$-level. WLOG $\{1\}$.

Take 2-element sets

$\{1,2\}, \ldots$ ~~If $\exists$ 2 not at $\mathbb{Z}$-level~~ they ~~must intersect~~ at $\{1\}$. ~~Anyone else {1} must be at ... level isect it with any~~

OK

$\{1,2\}$ $\{1,3\}, \ldots, \{1,n\}$. But then only one of these escapes $\{1\}$-level

Know what it does on full set and each full $- \{i\}$ determines all??

Can we recover the function from here? And is it always recoverable!

Yes, $f(\{1,2,3\}) = f([n]-\{4\}) \cap ([n]-\{5\}) \cap \cdots \cap ([n]-\{n\}))$

$f(\{1,\ldots,n-2\}) = f(n-1 \cap n-2) = \min \ldots$ ok know

Is it well defined? $f(S) = \min\limits_{i \notin S} f([n]-\{i\})$ Yes.

and if S = full, take $f([n])$

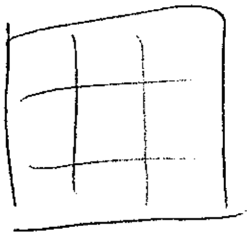
Try $f(S \cap T) = \min\limits_{\substack{i \notin S \\ \text{or} \\ i \notin T}} f([n]-\{i\})$

Verify condition OK.

vs $\min\limits_{i \notin S} f([n]-\{i\}) \ldots$ OK.

2002/A4

2011-09-18
1



zero determinant iff all 0 in row.

How to get 0 det with 5 1s, 4 0s

Two equal rows.

or third is sum of 2.

All will be of form coeff 1.

$c = a+b$?, with 5 1s?

impossible

So only way is 2 rows equal, or 1 row 0

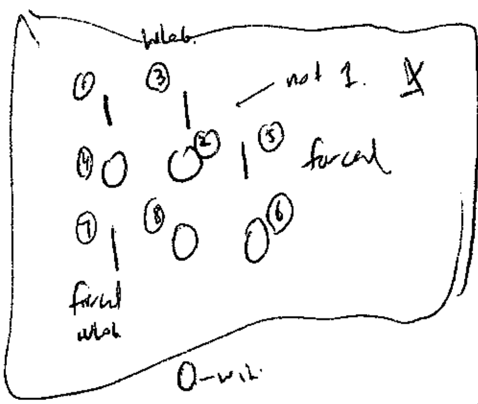
Player 1 can prevent 0 from 3 0s.

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

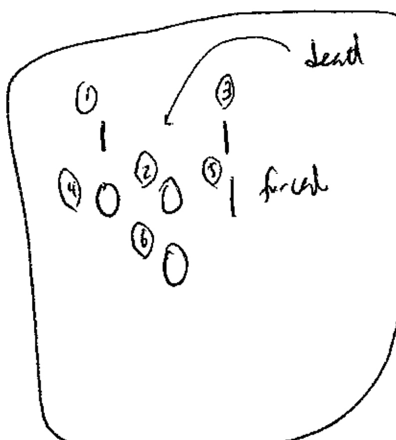
$$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \text{ equal}$$
$$\begin{matrix} 0 & 0 & 1 \end{matrix}$$

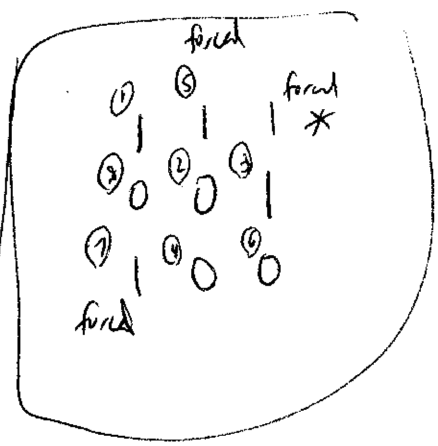
$$\begin{matrix} 1 & 1 & 1 \end{matrix}$$
$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{matrix} \text{ equal.}$$

$$\begin{matrix} 1 & 1 & \\ 0 & 0 & 1 \\ 1 & & 0 \end{matrix}$$
✓

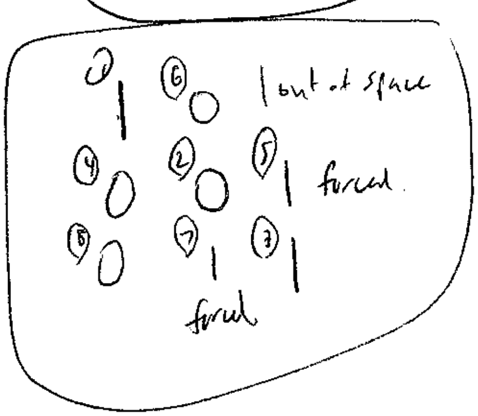


0-win.





0-wins!



2011-09-18

2002/A5

X X X X X X X X X ⊘ X

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1
✓	✓	✓	✓	✓	✓	✓									

symmetric OK.

Clearly if have $\boxed{1 \quad n}$ $\xrightarrow{\text{later}}$ $\boxed{1 \quad m}$.

$\boxed{2 \quad 3} \longrightarrow \boxed{2 \quad 5}$.

2^4-2 gets 2^4-1, ..

4 1

$\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$

$\frac{2}{1}$ $\frac{2}{3}$ $\frac{2}{5}$

$\frac{3}{1}$ $\frac{3}{2}$ $\frac{3}{4}$ $\frac{3}{5}$

$\frac{4}{1}$ $\frac{4}{3}$

$\frac{5}{2}$ $\frac{5}{3}$

$1 \quad n \to 1$ with n

Say want $4/5$. 4 5 1 so need $4 \quad 1 \leftarrow 3 \; 4 \; 1$

So need $3 \quad 1 \leftarrow 2 \; 3 \; 1$

So need $2 \; 1 \leftarrow 1 \; 2 \; 1$

So need $1 \; 1$ ✓.

In general to get

a/b with $a>b$,

(Never $a=b$) just need ~~ab~~ $a-b$, b. and not equal, else had gcd. ✓

else can reduce.

2002/A6

2011-9-18

(3)

$f(n) = n \cdot f(\log_b n)$

$f(1)=1$

$f(2)=2$.

$\underline{b=10}$: $n = 1 \ldots 9$ we do $f(n) = n \times f(1) = n$.

$n = 10 \ldots 99$ we do $f(n) = n \times f(2) = 2n$. For base 3+.

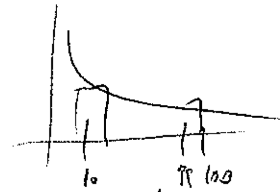
$n = 100 \ldots 999$ we do $f(n) = n \times f(3) = 3n$.

sum across $\frac{1}{f(100)} \cdots \frac{1}{f(999)} = \frac{1}{f(3)} \times [\log 999 - \log 100]$.

$\frac{1}{f(2)}[\log 100 - \log 10] + \frac{1}{f(3)}[\log 1000 - \log 100] + \frac{1}{f(4)}[\log 10000 - \log 1000]$

$= (\log 100)\left[\frac{1}{f(2)} - \frac{1}{f(3)}\right] + (\log 1000)\left[\frac{1}{f(3)} - \frac{1}{f(4)}\right] + \cdots$

$= c \cdot \left[\frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \cdots\right]$



For base $\underline{3+}$: $f(2)=2$ makes sense.

$n = 1 \ldots 9$: $f(n) = n \times f(1)$. so $\frac{1}{f(1)}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9}\right]$

$\int_{10}^{100} \frac{1}{x}\,dx = \ln 100 - \ln 10 = \ln 10$.

$n = 10 \ldots 99$: $f(n) = n \times f(2)$ $+ \frac{1}{f(2)}\left[\frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{99}\right]$

$+ \ldots$

it's $< \int_9^{99} \frac{1}{x}$.

$> \frac{1}{f(1)}[\ln 10] + \frac{1}{f(2)}[\ln 10] + \cdots + \cdots$

$\frac{1}{f(1)} + \cdots$

$\geq (\ln 10)\left[\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots\right]$ ✱.

$\log_3 4 = 2. + 1.$

$\log_3 5 = 2.x$

In fact for even 3 it fails.

So try $b=2$. Why does it converge? $f(n) = n \times \cancel{f(\log n)} f(\lfloor \lg n \rfloor + 1)$

$= n \times (\lfloor \lg n \rfloor + 1) \times \lg + 1$ of it, etc, until get to 2.

2011-09-18

4

So $\frac{1}{f(1)} + \cdots + \frac{1}{f(1023)} \leq 0.9 \times \left[\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(10)}\right]$.

really?

No.

$\frac{1}{f(1)}$

$\frac{1}{f(2)} + \frac{1}{f(3)} = \frac{1}{f(2)}\left[1 + \frac{1}{3}\right]$. $= \frac{1}{2 \cdot f(2)} + \frac{1}{f(2)}\left[\frac{1}{2} + \frac{1}{3}\right]$ $\frac{5}{6}$

$\frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} = \frac{1}{f(3)}\left[\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right]$ $\leftarrow$ $\ln \frac{7}{3}$

$\frac{1}{f(8)} + \cdots + \frac{1}{f(15)} < 0.9 \frac{1}{f(4)}$ $\quad \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15}$.

~~So $\frac{1}{f(4)} + \cdots + \frac{1}{f(1023)} < 0.9 \times \left(\frac{1}{f(3)} + \frac{1}{f(4)} + \cdots + \frac{1}{f(10)}\right)$,~~

So $\frac{1}{f(2)} + \cdots + \frac{1}{f(1023)} < \frac{1}{4} + 0.9\left(\frac{1}{f(2)} + \cdots + \frac{1}{f(10)}\right)$,

So $0.1\left[\frac{1}{f(2)} + \cdots + \frac{1}{f(1023)}\right] < 0.1\left[\frac{1}{f(2)} + \cdots + \frac{1}{f(10)}\right] + \left[\frac{1}{f(11)} + \cdots + \frac{1}{f(1023)}\right]$

$< \frac{1}{4}$

$\frac{1}{f(2)} + \cdots + \frac{1}{f(1023)} < \frac{10}{4}$ always.

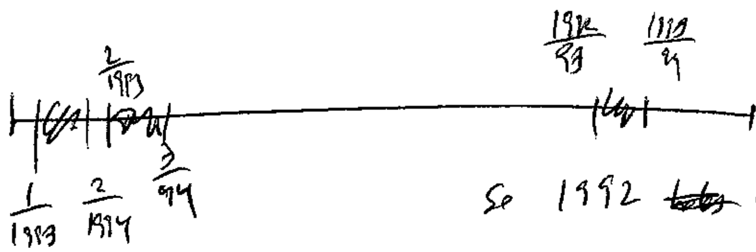
And carry this on, so it is bdd ✓.

Putnam ~~26~~ 1993 / B1

2011-09-19 (A)

$\frac{1}{1993} \ldots \frac{2}{1994}$. $\leftarrow \frac{1}{1992}$? will do $\frac{2}{1998}$?

$\frac{1992}{1993} \ldots \frac{1993}{1994}$ $\frac{1991}{1992}$ is now too small $\frac{1993}{1995} = 1 - \frac{2}{1995}$ vs $1 - \frac{1}{1993}$



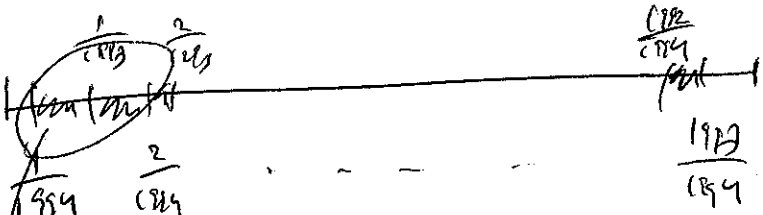
So 1992 ~~bets~~ windows inside

$\frac{1}{1993}$ $\frac{k}{n}$ $\frac{2}{1994}$ also before $1 - \frac{1}{1993}$ and $1 - \frac{1}{1994}$

So need before $\frac{1}{1994}$. $\frac{1}{1993}$, $\frac{2}{1994}$ In general, before $\frac{i}{1993}$ $\frac{i+1}{1994}$.

$\frac{1993-i}{1993}$ $\frac{1994-i}{1994}$

$1 - \frac{i}{1993}$ $1 - \frac{i}{1994}$.



After ... $\frac{1}{1994} \ldots \frac{1992}{1994}$ Total of $1992 \times 2 = 3984$ intermediates

after $\frac{1}{1993} \ldots \frac{1992}{1993}$ $\rightarrow$ try $3985. = 1992 \times 2 + 1$

Next

$$\frac{2i}{1992 \times 2 + 1} > \frac{i}{1993}.$$

$$2 \times 1993 i > 2 \times 1992 i + i$$

$$i > 0.$$

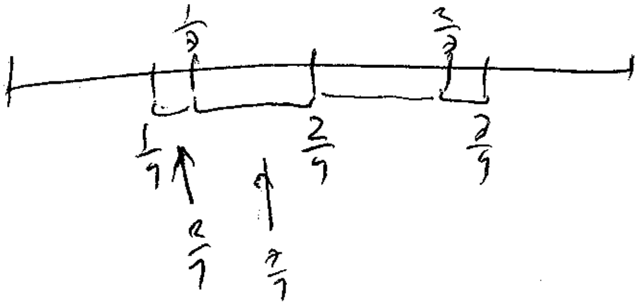
$$\frac{2i}{1992 \times 2+1} \text{ vs } \frac{i+1}{1994}$$

$$2 \times 1994 i \text{ vs } 2 \times 1992 i + i + 1992 \times 2 + 1.$$

$$3i \text{ vs } 1992 \times 2 + 1$$

2011-08-19
(8)

T7 $\frac{1}{3}, \frac{1}{4}$



7.

So $\dfrac{2i+1}{1993\times 2+1}$ vs $\dfrac{i}{1993}$ OK

$1993\times 2i + 1993 > 1993\times 2i + i$

$\dfrac{2i+1}{1993\times 2+1}$ vs $\dfrac{i+1}{1994}$

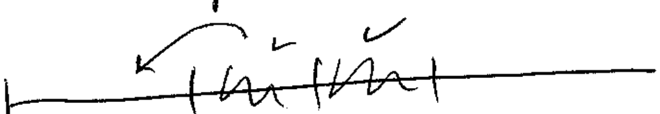
$1994\times 2i + 1994$ vs $(1993\times 2+1)i + (1993\times 2+1)$

i vs $1993\times 2+1 - 1993 - 1$

$<$ $= 1993$ OK

So we take $1993\times 2+1$.

Note: since we need representation between

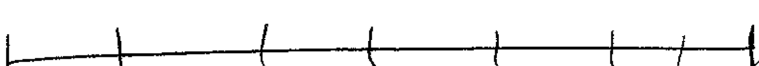


here, since already get 2 between $\frac{1}{1994} \dots \frac{2}{1994}$,

So $\Delta < \frac{1}{1994}$

Similarly need one at end.

So we took:



2×1993 pieces

the $\frac{1}{n} \dots \frac{n-1}{n}$ needs to be in the pieces $\Rightarrow (+1)$

cut with $\frac{1}{1993} \dots \frac{1992}{1993}$ — $1 + 1992$ pieces

then with $\frac{1}{1994} \dots \frac{1993}{1994}$ $+ 1993$ pieces

2011-09-19 ©

1993/B2

Perfect information. Cards 1..2n

A: x (n) B: $-x$ (n)

Sum is 0 mod $2n+1$.

? pairs make that sum

$n=1$: A: 1, B: 2. B wins

$n=2$: A: 2, 1; B: 3, 4. B wins

A: 4, 1; B: 2, 3. B also wins

$n=3$: A: $-x$, x; B: $-y$, y. A: x, B: y.

A, B, B's turn

Sum: k left, $k+1$ left

Sum nonzero.

B must avoid putting down someone who lets A instantly win. So must avoid $\underline{k}$ residues. But he has $k+1$ so he can.

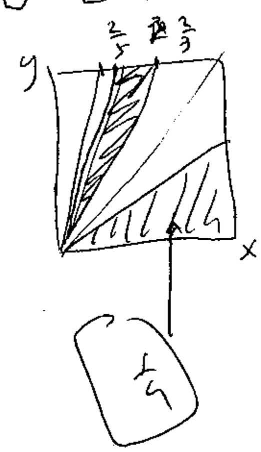
So B never loses.

k, $k+1$

not 0.

so B can play someone to avoid sum 0.

1993/A3



$\frac{x}{y}$ even. so between 0..0.5 or 1.5..2.5 or 3.5..4.5.

$\frac{2}{9}$ $\frac{2}{7}$ $\frac{2}{5}$ $\frac{2}{3}$

So want: $\frac{1}{2}\times$

$$\frac{1}{2}\times\left[\frac{2}{3}-\frac{2}{5}+\frac{2}{7}-\frac{2}{9}+\frac{2}{11}-\frac{2}{13}\cdots\right]$$

$$=\left[\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\right]$$

$$1-\frac{\pi}{4}+\boxed{\frac{1}{4}}=\frac{5}{4}-\frac{1}{4}\pi$$

$\frac{\pi}{2}=\cos^{-1}0.$

$\cos^{-1}x$:

Taylor for

$$2\times\left(\frac{1}{15}+\frac{1}{63}+\frac{1}{143}+\cdots\right]$$

der: $\frac{-1}{\sqrt{1-x^2}}$ at $x=0$: -1

der2: $\frac{2x}{2(1-x^2)^{\frac{3}{2}}}$ at $x=0$: 0.

der3:

$f(x)=\cos^{-1}x$.

$\cos f(x)=x$

$(-\sin f(x))f'(x)=1$.

$f'(x)=-\frac{1}{\sin\cos^{-1}x}$

hyp 1, $\sqrt{1-x^2}$, adj x, θ

x, 1

$\tan^{-1}x$: $\frac{1}{1+x^2}$

so $\int_0^1\frac{1}{1+x^2}dx=\tan^{-1}1=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots$

IMO 1993/1

2011-09-20
(A)

$f(x) = x^n + 5x^{n-1} + 3.$

$(x^a + bx^{a-1} + (\ ) + 1)\ (x^c + dx^{c-1} + (\ ) + 3)$

$a + c = n.$
$b + d = 5.$

$$\begin{array}{ccc} 1 & b & e \\ 1 & c & f \end{array}$$

$e + f + bc = 0.$

$1 + a_1x + a_2x^2 + a_3x^3$

$3 + 3a_1x + \boxed{+\underset{3a_1^2 - 3a_2}{a}x^2} + cx^3$

needs 3. for making the x^3 term

$3a_2 - 3a_1^2 + c = 0.$

$1 + a_1x + a_2x^2 + a_3x^3 + \dots + a_rx^r$ ~~If s<r~~

$3 + b_1x + b_2x^2 + \dots + b_sx^s.$

or up to the x^s term, what can we say? Say $r \geq 2$ then all up to x^s are 0

Induction: if $b_1 \equiv 0 \equiv b_2 \equiv \dots \equiv b_k \pmod 3$

then for x^{k+1} term, get $b_{k+1} + 3(\ ) \equiv 0$, so $b_{k+1} \equiv 0 \pmod 3$

But poly not div by 3.

Now check: if $r = 1$. Then have $(1+x)$ or $(1-x)$. must be monic.

$(\qquad)$.

i.e. -1 is root. but $(-1)^n + 5(-1)^{n-1} + 3$ is never 0.

USAMO 1995/4

2011-09-20 (B)

WLOG, $a_0 = 0$.

Now $m \mid a_m$ for all m.

But polynomially bounded.

Use $P(n)$ to do Lagrange interpolation?

Degree is d.

$2 \mid a_5 - a_3$

so periodic mod any prime

For each prime,

So take $\frac{a_n}{n} = a_n^{(1)}$ still is integer, bdd by one smaller polynomial ...

Check: $a_n^{(1)} - a_m^{(1)} = \frac{a_n}{n} - \frac{a_m}{m}$.

Div by $n-m$?

$a_n = a_m + k(n-m)$ $\rightarrow$ $\underbrace{a_m - km}_{\text{div by } m} = \underbrace{a_n + nk}_{\text{div by } n}$.

So $\frac{a_m + k(n-m)}{n} - \frac{a_m}{m} = a_m\left(\frac{1}{n} - \frac{1}{m}\right) + k\left(1 - \frac{m}{n}\right)$

$= a_m\left(\frac{m-n}{mn}\right) + k\left(1 - \frac{m}{n}\right)$

How about derivative instead?

fine if gcd is 1 $= \frac{a_m(m-n) + km(n-m)}{mn}$

$= (n-m)\frac{a_m + km}{mn} = (n-m)\frac{\frac{a_m}{m} + k}{n}$.

And if gcd is not 1?

But $a_0^{(1)}$ is now in trouble since it was $\frac{0}{0}$. Can't divide.

Derivative?

Take $a_n - a_{n-1}$. Should be discrete derivative, so smaller poly nomial

$(m-n) \mid a_m - a_{m-1} - a_n + a_{n-1}$ Yes

So now have new sequence

is it now bdd by smaller degree??

Say was bdd by $P(n)$.

Or ~~put~~ assume it is within $\pm\Delta$ of a polynomial $P(n)$.

Now after discrete deriving, it's within $\pm 2\Delta$ of polynomial of lower degree

$\vdots$

entirely within $\pm 2^{\deg}\Delta$ of $\xleftrightarrow[\text{dim}]{}$.

2011-09-20

(E)

USAMO 1995/4

Try $P(n) = \text{const}$. Is it clear the q's are const?

Eventually the $m \mid q_m$ forces $q_m = 0$.

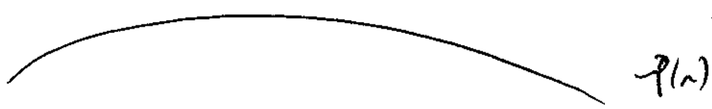
Then also $m-i \mid q_m - q_i = -q_i$, forces $q_i = 0$ too.

How much can $q_{k+1} - q_k$ be?

well, $q_{k+1} - q_M$ is div by $M-k-1$.

$q_k - q_M$ is div by $M-k$.

~~still~~ $\cancel{q_{k+1} - q_k}$ ~~is div~~



or $q_k - q_{k+q_k}$ is div by q_k

or $q_k - q_{k+q_k q_{k+1}}$

What if eventually monotone?

Or do $\dfrac{q_k - q_0}{k}$ and then re-index to throw away q_0 entirely.

Now have q_1 which we will handle down

And $\dfrac{q_k}{k} \le \dfrac{P(k)}{k} \mp \text{error}$

So check: $q_n = q_m + k(n-m)$ But $\dfrac{q_n}{n} - \dfrac{q_m}{m} = \dfrac{q_m m - q_m n + k m (n-m)}{mn}$

$q_n - kn = q_m - km$

$= (n-m) \dfrac{km - q_m}{mn}$

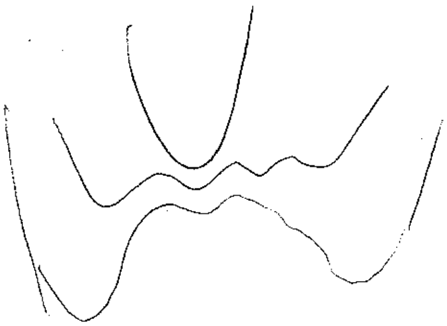
fine if $\gcd = 1$.

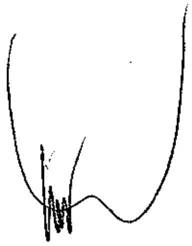
Now say

USAMO ~~1974~~ 2002/3

2011-09-20

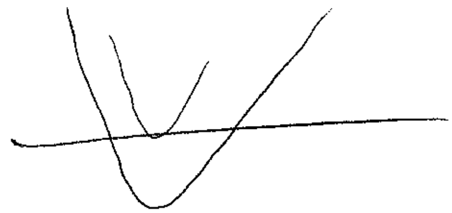
D





Will do.

Now make it monic.



GA 151 $\quad P(k)=\frac{k}{k+1}$ for $k=0\ldots n$.

degree n.

... too big

$$x(x-1)\cdots(x-n)\frac{0}{n!} + x(x-2)\cdots(x-n)\frac{\frac{1}{2}}{(n-1)!}$$

$n=1$: $\quad k=0,1$

$\quad 0 \;\; \frac{1}{2} = \frac{1}{2}x$

$n=2$: $\quad k=0,1,2$ $\qquad \cancel{x(x-2)=1}$

$\quad 0 \;\; \frac{1}{2} \;\; \frac{2}{3}$

$ax^2+bx = Q$ $\qquad \frac{1}{18}x^2+\frac{4}{9}x$

$a+b=\frac{1}{2}$

$4a+b=\frac{2}{3}$

$3a=\frac{1}{6}$

$a=\frac{1}{18}$

± 1 coeffs zeros circled

2011-09-20

GA 168

$P(z)=0 \Leftrightarrow Q(z)=0.$ So $P(z) = a(z-r_1)(z-r_2)\cdots(z-r_k)$

$P(z)=1 \Leftrightarrow Q(z)=1.$ $Q(z).$

$P(z)\,Q(z)$ has only same zero

$$a(z-r_1)(z-r_2)^2 = 1 + b(z-r_3)(z-r_4)^2$$

$$c(z-r_1)^2(z-r_2) = 1 + d(z-r_3)^2(z-r_4)$$

GA 183 $x^{p-1}+x^{p-2}+\cdots+x+1 = \dfrac{x^p-1}{x-1} \Rightarrow \dfrac{(x+1)^p-1}{x}$

$\downarrow$

$x+1$

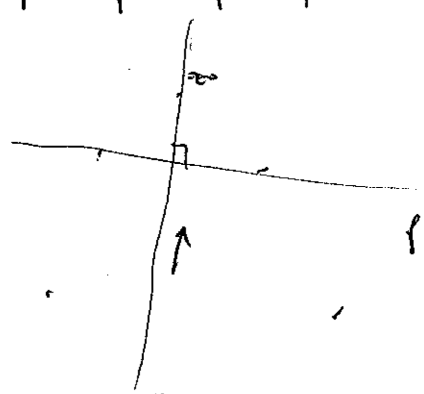
$$= x^p + p\,x^{p-1} + \binom{p}{2}x^{p-2} \cdots + px.$$

Question: $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots + a_0$

$np \quad np \quad np \quad np$ up to x^{n-1} coeff.

$p \quad p \quad p \quad p$

Lucas.



So all zeros lie below this line $a\,\mathrm{Re}\,z + b\,\mathrm{Im}\,z \le c.$

$P(z)$ zeros. Say $P'(z_0)=0$. Need it also here

Take $P(z)$.

VTRMC 2004/5 — 2011-09-29 (A)

$f(x) = \int_0^x \sin(\underbrace{t^2 - t + x}_{x \dots x^2})\,dt.$

$f'(x) = ?$ $\qquad t^2 - t = t(t-1),$ der: $2t-1$, incr. on $t > \frac{1}{2}$

$$f(x) = \int_0^x g(t,x)\,dt.$$

$$f'(x) = \lim_{h\to 0} \frac{\int_0^{x+h} g(t,x+h)\,dt - \int_0^x g(t,x)\,dt}{h}$$

$$\frac{\int_0^{x+h} g(t,x+h)\,dt - \int_0^{x+h} g(t,x)\,dt}{h} + \frac{\int_0^{x+h} g(t,x)\,dt - \int_0^{x} g(t,x)\,dt}{h}$$

$$= \lim_{h\to 0} \int_0^{x+h} \frac{g(t,x+h) - g(t,x)}{h}\,dt + g(x,x)$$ add since g_x def

$$= \int_0^x g_x(t,x)\,dt + g(t,x) + \lim_{h\to 0} \int_x^{x+h} \frac{g(t,x+h)-g(t,x)}{h}\,dt$$

width $\to 0$

So $f'(x) = \int_0^x \cos(t^2 - t + x)\,dt + \sin(x^2).$

$$f''(x) = \int_0^x -\sin(t^2 - t + x)\,dt + \cos(x^2) + \cos(x^2)\,2x.$$

$$f''(x) + f(x) = \cos(x^2)\left[2x+1\right] =$$

Now der 10 more times, then substitute 0.

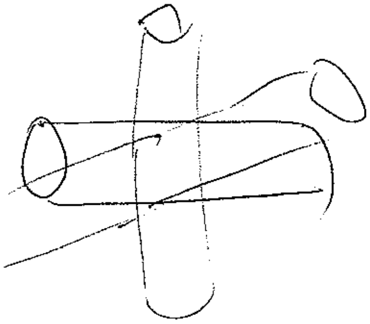
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \dots \Rightarrow \cos x^2 = 1 - \frac{x^4}{4!} + \frac{x^8}{8!} - \frac{x^{12}}{12!} \dots$$

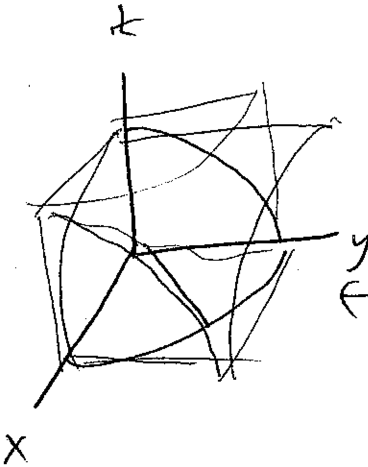
~~coeff of x^{10} after $*(2x+1)$ is $1 - \frac{1}{10!}$~~

$*(2x+1)$ has coeff of x^{10} to be $\underline{\underline{0}}$. $\square$

VTRMC 2001/1

2011-09-24 (6)





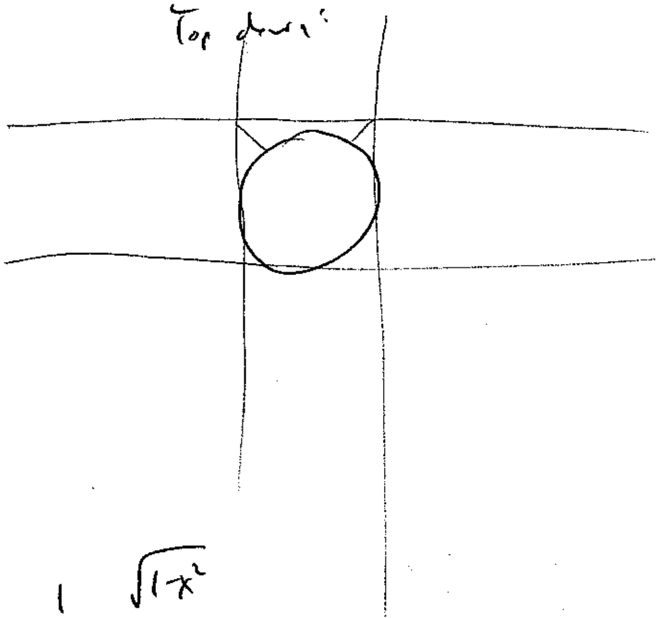
← looks like the sphere

Obviously, if $x^2+y^2+z^2 \le 1$, this is true

But say it's true.
Why is $x^2+y^2+z^2 \le 1$?
~~Avg this~~: *
Avg this: $2(x^2+y^2+z^2) \le 3$
$x^2+y^2+z^2 \le \frac{3}{2}$

Indeed can attained iff $x=y=z=c$, sticks out quite far

Top down:



$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}$$

Right. Say $x^2+y^2 \le 1$.

Now can we always pick z?

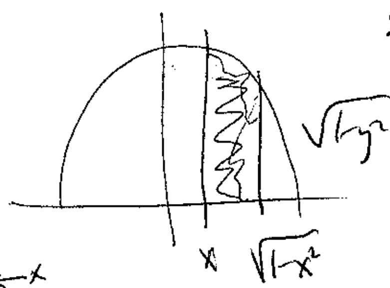
Need z st. $z^2+x^2 \le 1$ && $z^2+y^2 \le 1$.

So z up to $\sqrt{1-\max(x^2,y^2)}$

$\sin\phi = \cos\theta$

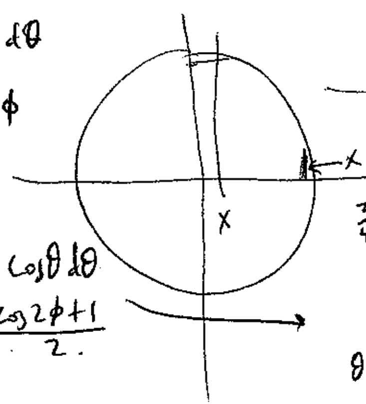
So do $8\times 2 \int_{x=0}^{1/\sqrt{2}}\int_{y=x}^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx.$

$\sin\phi = \sin\theta$



$x = \sin\theta$ $\quad dx = \cos\theta\,d\theta$

$y = \sin\phi.$ $\quad dy = \cos\phi\,d\phi$



$$\int_{\theta=0}^{\frac{\pi}{4}}\int_{\phi=\theta}^{\frac{\pi}{2}-\theta} \cos\phi \cdot \cos\phi\,d\phi\,\cos\theta\,d\theta$$

$\cos^2\phi = \frac{\cos 2\phi + 1}{2}.$

$$\int_{\theta=0}^{\frac{\pi}{4}} \frac{\left[\frac{1}{2}\sin(\pi-2\theta) - \frac{1}{2}\sin 2\theta\right] + \frac{\pi}{2} - 2\theta}{2}\cos\theta\,d\theta$$

$$= \int_{\theta=0}^{\pi/4}\left[\cancel{-\cos 2\theta} + \frac{\pi}{4} - \theta\right]\cos\theta\,d\theta$$

VTRMC 2001/1 2011-09-24

©

$$\int \theta\cos\theta\, d\theta = \theta\sin\theta - \int \sin\theta\, d\theta = \theta\sin\theta + \cos\theta + C.$$

So get:

$$\cancel{\int_{\theta=0}^{\frac{\pi}{4}} -\cos 2\theta\cos\theta\, d\theta} + \underbrace{\frac{\pi}{4}\Big[\sin\theta\Big]_0^{\frac{\pi}{4}}}_{\frac{\pi}{4}\frac{\sqrt2}{2}} - \Big[\theta\sin\theta + \cos\theta\Big]_0^{\frac{\pi}{4}}$$

$$\underbrace{\frac{\pi}{4}\frac{\sqrt2}{2} - \left[\frac{\pi}{4}\frac{\sqrt2}{2} + \frac{\sqrt2}{2} - 1\right]}_{1-\frac{\sqrt2}{2}}.$$

$$\cancel{-\cos 2\theta\cos\theta = \cos(2\theta + \theta) - \cos(2\theta - \theta)}$$

$$\cancel{\cos 2\theta\cos\theta - \sin 2\theta\sin\theta - \cos 2\theta\cos\theta + \sin 2\theta\sin\theta}$$

$$\cancel{= \cos 3\theta - \cos\theta.}$$

$$\cancel{\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1.}$$

So just $8 \times 2 \times \left[1 - \frac{\sqrt2}{2}\right] = 16 - 8\sqrt2$

VTRMC 2000/3.

$$\left.\begin{array}{l} y' = y^2 - t^2. \\ y(0) = 0. \end{array}\right\} \Rightarrow \lim_{t\to\infty} y'(t) \text{ exists}$$

Why doesn't it blow up?

~~If $|y| \le t$, then $y' \le 0$.~~

If $0 \le y \le t$, then $y' \le 0$, so it decreases. This pushes it away from reaching t.

But if $y < -t$, then it increases so it never stays below $-t$.

Let $z = t - y$. Want $z \ge 0$ always.

Then $z' = 1 - y' = 1 - (y+t)(y-t) = 1 + (y+t)z = 1 + z(2t - z)$

Or how about $z = t^2 - y^2$. Would like always $z \ge 0$.

$$z' = 2t - 2yy' = 2t - 2y^3 + 2yt^2 = 2t + 2yz.$$

2011-09-24
D

$y' = y^2 - t^2$

$(t-y)' = 1 - y^2 + t^2 = 1 + (t-y)(t+y)$

~~$(t+y)' = 1 + y^2 - t^2 = 1+$~~

$(t+y)' = 1 + y^2 - t^2 = 1 - (t-y)(t+y)$

Guess: -1

If it stabilizes at $y' = c$.

Then $y = a + ct$. so $c = (a+ct)^2 - t^2 = a^2 + 2act + (c-1)t^2$

No ~~so $a=0$ and $c=1$~~

Test $(y+t)^2 = z$.

$z' = 2(y+t)(1+y^2-t^2)$

$z = \frac{y}{t}$.

Could be $\ominus$

$z' = \frac{ty' - y}{t^2} = \frac{1}{t}\big[\underbrace{y^2 - t^2}_{\text{always} \le 0}\big] - \frac{y}{t^2} = \frac{y^2}{t} - t - \frac{y}{t^2}$

~~So z' always~~

Want $y^2 - t^2 \to -1$.

$y^2 \to t^2 - 1$.

$y \to -\sqrt{t^2-1} = -t + \frac{1}{2t}$

If $y^2 - t^2 < -2$, then we catch up to the lower envelope

i.e. $y > -\sqrt{t^2-2} \approx -t + \frac{1}{t}$

so y is always $< -t + \frac{1}{t}$,

$y'' = 2y(y^2 - t^2) - 2t$

$\ominus \quad \ominus$

Putnam 2005/A5

2011-09-25 (A)

$$1 + \cancel{\tan^2\theta} = \cancel{\sec^2\theta} \ ?$$

$$\cos^2\theta + \sin^2\theta = 1$$

$x = \tan\theta$

$$\text{deriv } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta} = \sec^2\theta$$

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

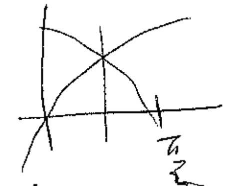
$$= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$$

$$\int \frac{\theta}{\cos^2\theta + \sin\theta}\,d\theta$$

$$\int \log(1+\tan\theta)\,d\theta = \log(1+\tan\theta)\,\theta - \int \frac{\sec^2\theta}{1+\tan\theta}\,\theta\,d\theta$$

$$\int \log\frac{1}{1+\tan\theta}\,d\theta = \left(\log\frac{1}{1+\tan\theta}\right)\theta - \int (1+\tan\theta)\cancel{\sec^2\theta} \times \theta\,d\theta$$

$\log(\sin\theta + \cos\theta)$



$$\frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$

$$\cancel{\theta} \quad \cancel{\int \theta\sec^2\theta} + \int \theta\frac{\sin\theta}{\cos^3\theta}$$

$$\int \theta\sec^2\theta = \theta\tan\theta - \int \tan\theta\,d\theta$$

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta = -\log\cos\theta + C$$

$$\int \theta\frac{\sin\theta}{\cos^3\theta}\,d\theta = \theta\frac{1}{2\cos^2\theta} - \int \frac{1}{2\cos^2\theta}\,d\theta = \frac{\theta}{2\cos^2\theta} - \frac{\tan\theta}{2}$$

So $\displaystyle\int \log(1+\tan\theta)\,d\theta = -\int \log\frac{1}{1+\tan\theta}\,d\theta = \theta\log(1+\tan\theta) + \theta\tan\theta + \log\cos\theta + \frac{\theta}{2\cos^2\theta} - \frac{\tan\theta}{2}$

At $\theta = \frac{\pi}{4}$: $\quad \frac{\pi}{4}\log 2 + \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} + \frac{\pi}{8 \times \frac{1}{2}} - \frac{1}{2}$

At $\theta = 0$: $\quad 0 + 0 + 0 + 0 - 0$

So diff is: $\quad \frac{\pi}{4}\log 2 + \frac{\pi}{4} - \frac{1}{2}\log 2 + \frac{\pi}{4} - \frac{1}{2}$

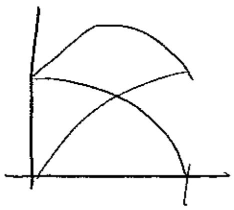
$$= \frac{\pi}{4}\log 2 + \frac{\pi}{2} - \frac{1}{2}\log 2 - \frac{1}{2}.$$

2011-09-28

A

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta = \cancel{\int \log(1+\tan\theta)}$$

$1-\tan^2\theta = 2-\sec^2\theta$



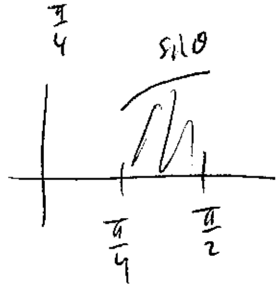
$\log(\underbrace{\sin\theta+\cos\theta}) - \log\cos\theta$

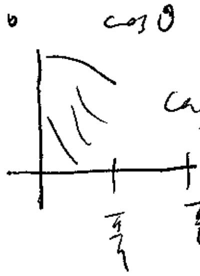
$= \sqrt{2}\,\sin\left(\theta+\frac{\pi}{4}\right)$?

$\sqrt{2}\,\sin\theta\,\frac{1}{\sqrt{2}} + \sqrt{2}\cos\theta\,\frac{1}{\sqrt{2}}$ ✓

so $\underbrace{\int_0^{\frac{\pi}{4}} \log[\sqrt{2}]}_{\frac{1}{2}\frac{\pi}{4}\log 2} + \int_0^{\frac{\pi}{4}} \log\sin\left(\theta+\frac{\pi}{4}\right) - \int_0^{\pi/4} \log\cos\theta$

$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\sin\theta - \int_0^{\frac{\pi}{4}} \log\cos\theta$





cancel

$\square$

1992/A1 $f(n) = 1-n$ $(\mathbb{Z})$ 2011-09-25

① $f(f(n)) = n$ ⟵ own inverse ⇒ bijection with fixed pts

② $f(f(n+2)+2) = n$

③ $f(0) = 1$

So $f(1) = 0$.

Say ~~f(f(c))~~ $f(c) = c$ Then $f(c+2) = c-2$

② inverse: $f(n+2)+2 = f(n)$ f again to ②

$f(n+2) = f(n) - 2$.

Use $f(0) = 1$. Then $f(2) = 1-2 = -1$.

$f(4) = -1-2 = -3$

induction to all positive evens

$1 = f(0) = f(-2) - 2$

$3 = f(-2)$ ... induction all negative evens

And use $f(1)$ to get all ⊕/⊖ odds.

1992/A2. $C(\alpha)$ is coeff of x^{1992} in pow series of $(1+x)^{\alpha}$.

so $\binom{\alpha}{1992}$.

$$\int_0^1 \underbrace{C(-y-1)}_{-1\ldots-2} \sum_{k=1}^{1992} \frac{1}{y+k}\, dy$$

$$= \int_0^1 \binom{-y-1}{1992} \sum_{k=1}^{1992} \frac{1}{y+k}\, dy$$

$$\frac{(-y-1)(-y-2)(-y-3)\cdots(-y-1992)}{1992!} \cancel{\sum_{k=1}^{1992}} \left(\frac{1}{y+1} + \frac{1}{y+2} + \cdots + \frac{1}{y+1992}\right)$$

$= \frac{1}{1992!} \times$ derivative of $(y+1)\cdots(y+1992)$

$$= \frac{1}{1992!} \times \left[(2\cdot3\cdot4\cdots1993) - (1\cdot2\cdots1992)\right]$$

$$= \frac{1}{1992!} \times \left[1993 \times 1992! - 1992!\right]$$

$$= 1992.$$

2011-09-25

D

1992/A3 $\mathbb{Z}^+$: $(n,m)=1$: $(x^2+y^2)^m = (xy)^n$

Given m

$x^2+y^2 = xy$? $(x+y)^2 = 3xy$. so $3 \mid x+y \Rightarrow 9 \mid LHS \Rightarrow 3 \mid x$ or $y \Rightarrow 3 \mid$ both x,y

$(x-y)^2 = -xy$. Infinite descent.. $\divideontimes$

$(x^2+y^2)^m = xy$?

Say $m=1$: $x^2+y^2 = (xy)^2$?

$A+B = AB$.

$1 = AB-A-B+1$

$1 = (A-1)(B-1)$ so $A,B=2$. But not squares $\divideontimes$

$x^2+y^2 = (xy)^3$? WLOG $x \le y$.

$x^2+y^2 \le 2y^2 \le y^3$ if $y \ge 2$. Only will work if $(x,y)=(1,1)$ or $(1,2)$ both fail

$x^2+y^2 = (xy)^4$? ~~Same~~ worse problem.

Say $m=2$. $(x^2+y^2)^2 = xy$.

Actually, $(x^2+y^2)^m = (xy)^n \Rightarrow x \mid RHS \Rightarrow x \mid (x^2+y^2)^m$ ~~&xy~~ $\Rightarrow x \mid y^{2m}$.

$= x^{2m} + \dots + m x^2 y^{2(m-1)} + y^{2m}$

So $x \mid y^{2m}$, $y \mid x^{2m}$.

$x=y$: $\underbrace{2^m x^{2m}}_{\text{odd \#2's}} = \underbrace{x^{2n}}_{\text{even \#2's}}$ → $2^m = x^{2(n-m)}$

$x=2^k$. So $m = 2(n-m)k$

$k = \frac{m}{2(n-m)}$

no factor with m. But $k \in \mathbb{Z}$, so must be exactly 1.

Get $2^{\frac{m}{2}}$.

Say $x = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t+m_t}$ common

$y = p_1^{b_1} \cdots p_t^{b_t+m_t}$

WLOG $x \le y$ Now y

Now scale out by gcd

$$\left[p_1^{m_1} \cdots p_t^{m_t}\right]^{2m} \left((x')^2+(y')^2\right)^m = [gcd]^{2n}(x'y')^n.$$

Case 1: $m<n$. ~~Then extra $\forall i$ p_i on RHS.~~

Now suppose have p_i in x' or y'. Then forced p_i in both ($\divideontimes$ gcd)

So $x'=y'=1 \Rightarrow x=y$, impossible

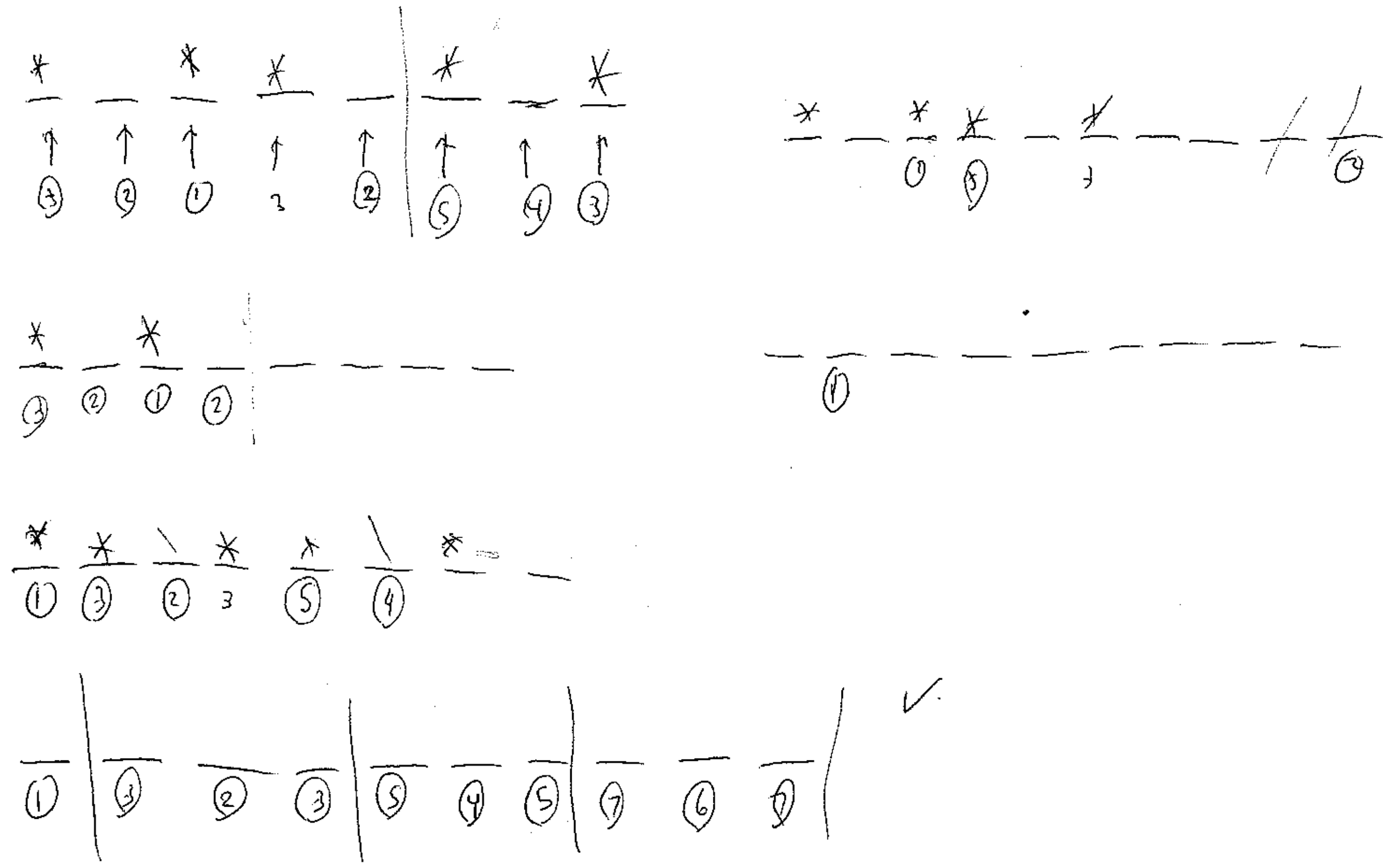
Case 2: $m>n$. Then $\forall i$ in gcd, have extra $p_i^{2(m-n)}$ in LHS.

Only primes p_i appear in RHS, and they all appear in x' or y'.

But $(x')^2+(y')^2$ is not divisible by any of those primes, $\divideontimes$.

So it generates new primes.

2002/B4 2011-10-02 (A)



2002/B5

$| * |$

$b^2 + 1 + xb = N$

$\Rightarrow x = \frac{N - b^2 - 1}{b} = \frac{N-1}{b} - b.$ Need solvable to be in $\{0, \dots, b-1\}$ for 2002 values.

So $\frac{N-1}{b} \in \{b, b+1, \dots, 2b-1\}$.

Need $\frac{N-1}{b} \geq b$
$\sqrt{N-1} \geq b$
and $\frac{N-1}{b} < 2b$, i.e. $b > \sqrt{\frac{N-1}{2}}$.

So b is from $\sqrt{\frac{N-1}{2}} \dots \sqrt{N-1}$
and factor of $N-1$.

Just find number N s.t. 2002 ints between $\sqrt{\frac{N}{2}}, \dots, \sqrt{N}$ are factors of $N-1$.

$N-1 = M!$ Assemble $\log 2, \log 3, \dots, \log M$ to get between $\frac{\Sigma}{2}$ and $\frac{\Sigma}{2} - \frac{\log 2}{2}$.

Wlt if $\log 2 + \dots + \log K$ is $\frac{\Sigma}{2}$

or $\log(K+1) + \dots + \log M$ is $\frac{\Sigma}{2}$

Now flip $\log(K+2002) \nearrow \log K$ choose by tiny.

So need $\log 2 + \dots + \log K$ within $\frac{1}{10}$ of $\log 2 + \dots + \log M$

Take $\times 10000 \times \underbrace{10001 \times 10002 \times \dots \times 12002}_{\sqrt{N}}$

Hm, take $N-1 =$

10001	10000 *
10002	10002
⋮	⋮
12002	12002

Now any substitute of 10000 into column (1) makes product $< \sqrt{N-1}$, but not by much.

2002 B6

2011-10-02 (B)

$\cancel{(x-y)^{p-1} \overset{?}{=} x^{p-1} - y^{p-1}}$

~~well~~ $\cancel{(x-y)^p = x^p - y^p \cdot (x-y)(}$

$x^{p-1} \equiv 1 \pmod p$

$$\begin{vmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{vmatrix} = \begin{vmatrix} x^{p^2} & y^{p^2} & z^{p^2} \\ 0 & y^p & \\ & & \end{vmatrix}$$

$p=3$: $x^2 - y^2$ easy

$p=5$: $x^4 - y^4 \ldots x^2+y^2 \mod 5$

$(x^4-y^4)^5 = x^{20} - y^{20}$

$(x+3y)(x+2y) = x^2+y^2$

$$= \begin{vmatrix} x & y & z \\ 0 & y^p - x^{p-1}y & z^p - x^{p-1}z \\ 0 & y^{p^2} - x^{p^2-1}y & z^{p^2} - x^{p^2-1}z \end{vmatrix} = x\cdot\Big[y^p z^{p^2} - y^p x^{p^2-1} z - x^{p-1} y z^{p^2} + x^{p^2+p-2} yz$$

$$- y^{p^2} z^p + y^{p^2} x^{p-1} z + x^{p^2-1} y z^p - x^{p^2+p-2} yz$$

only need this guy. Factor out extra y's, z's, x's

$$\begin{vmatrix} y^{p-1} - x^{p-1} & z^{p-1} - x^{p-1} \\ y^{p^2-1} - x^{p^2-1} & z^{p^2-1} - x^{p^2-1} \end{vmatrix}.$$

$a - b$

$a^{p+1} - b^{p+1} = \cancel{a^{p+1} + a^p +}$

$$= z^{p^2-1}\left[y^{p-1} - x^{p-1}\right] - x^{p^2-1}\left[y^{p-1} - z^{p-1}\right] - y^{p^2-1}\left[z^{p-1} - x^{p-1}\right]$$

$$= C^{p+1}[B-A] + A^{p+1}[C-B] + B^{p+1}[A-C].$$

Lucas: $\binom{p+1}{1} \bmod p = \binom{1}{0}\binom{1}{1} \bmod p = 1$

$\binom{p+1}{2} \bmod p = 0$

$\binom{p+1}{p-1} \bmod p = 0$

$\binom{p+1}{p} \bmod p = \binom{1}{1}\binom{1}{0} = 1.$

UFD so must be able to factor $y^{p-1} - x^{p-1}$!

$(x+1)^p = x^p + 1.$

Then

$$\begin{vmatrix} 1 & 1 \\ y^{(p-1)p} + y^{(p-1)(p-1)} x^{p-1} + y^{(p-1)(p-2)} x^{(p-1)2} + \cdots + x^{(p-1)p} & z^{(p-1)p} + z^{(p-1)(p-1)} x^{p-1} \cdots \end{vmatrix}$$

Diff: $z^{(p-1)p} - y^{(p-1)p} + \left[z^{(p-1)(p-1)} - y^{(p-1)(p-1)}\right] x^{p-1} + \left[z^{(p-1)(p-2)} - y^{(p-1)(p-2)}\right] x^{(p-1)2} + \cdots$

$+ \left[z^{p-1} - y^{p-1}\right] x^{(p-1)(p-1)}$

Factor out $z^{p-1} - y^{p-1}$

$$= \sum_{a+b+c=p-1} (x^{p-1})^a (y^{p-1})^b (z^{p-1})^c.$$

2002 IB6 2011-10-02 ©

Use: $(x+y)(x+2y)(x+3y)\cdots(x+(p-1)y) = x^{p-1}-y^{p-1}$.

Now to get: e.g. $p=3$.

$$(x^2)^2+(y^2)^2+(z^2)^2+(x^2)(y^2)+(x^2)(z^2)+(y^2)(z^2)$$ it's 6 terms

$\cancel{(x^2-y^2+z^2)(}$ $(x+y+z)(x+y+z)(x-y+z)(x+y-z)$

$(x^2+y^2+z^2)^2$

$(x+y)(y+z)(z+x)$ has x^2y terms, etc

$6=3\times2\times1$.

mod 3

$a^2+b^2+c^2+ab+bc+ca \equiv a^2+b^2+c^2-2ab-2bc-2ca$.

$(a+b-c)^2 = a^2+b^2+c^2+2ab-2bc-2ca$. ~~(ka+kb+kc)^2~~

$(ka+kb+kc)^2$ No. it will factor through

$(a\pm b\mp c)^2$ ~~problem is i~~ solve

$(xa+yb+zc)^2$. Need $2y=1$ $y=-1$.

$2z=1$.

$2yz=1$.

(a

$2y=2z=2yz$.

$(x^2+y^2-z^2)^2 = x^4+y^4+z^4+2x^2y^2\underbrace{-2y^2z^2}_{y^2z^2}\underbrace{-2x^2z^2}_{y^2z^2}$ $y=z=yz$

$-x^2y^2$ does it $x=x^2$. only sol 0, 1.

so it's $[x^2+y^2-z^2+xy][x^2+y^2-z^2-xy] = (x+y+z)(x+y-z)$

Factor $x^2+xy+y^2-z^2$. it's $x^2-2xy+y^2-z^2 = (x-y)^2-z^2 = (x-y+z)(x-y-z)$

this gives for $p=3$.
4 ~~...~~ linear terms

2011-10-02 (D)

2002/B6

$p=3$: 4 terms: $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$.

Suggests $\prod_{a,b\in 1}^{p-1}(x+ay+bz)\ldots$

~~coeff of x~~
Note: $z^{(p-1)^2}=1$.

So try $a=1$: $\prod_b[(x+y)+bz] = (x+y)^{p-1}-z^{p-1}$.

$a=2$ $\prod_b((x+2y)+bz) = (x+2y)^{p-1}-z^{p-1}$

$\vdots$

So $\prod_{a=1}^{p-1}\left[(x+ay)^{p-1}-z^{p-1}\right] \Rightarrow$ ~~all~~ only terms are of form $x y$ $z^{\text{mult of } p-1}$

sum degs $(p-1)2$

Symmetry $\Rightarrow$ all trys have mult. ~~of p-1~~ and sum to desired

$= \underbrace{\prod_{a=1}^{p-1}(x+ay)^{p-1}}$ + swap some with z^{p-1}'s

this already gives $(x^{p-1}-y^{p-1})^{p-1}$

Remains to show correct coeffs

$x^{p-1}-y^{p-1}$

$(x-y)^{p-1} = x^{p-1}+x^{p-2}y+x^{p-3}y^2+\cdots+y^{p-1}$.

Good

Now coeff of term with z^{p-1}.. Say know $x^{0}y^{0}$ ← sum to $(p-1)t$.

Want to add: all choices of t guys from $\{1,2,\ldots,p-1\}$

take $(x+\_y)$ product. The $p-1$ powers

~~or day~~

are: $(x+y)^{p-1}+(x+2y)^{p-1}+(x+3y)^{p-1}+\cdots+(x+(p-1)y)^{p-1}$.

$\uparrow$

$=$ all pairs

1992/B1 2011-10-03 (A)

AP: $1, 2, 3, 4, 5, 6, \dots, n$ Sums: $\underset{3}{1+2}, \underset{4}{1+3}, \dots, n-1+n = 2n-1$

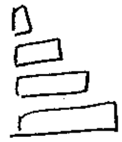
so $2n-3$

PF. $\underbrace{a_1+a_2 < a_1+a_3 < a_1+a_4 < \dots < a_1+a_n}_{n-1} < \underbrace{a_2+a_n < \dots < a_{n-1}+a_n}_{n-2}$ ok

1992/B2. How to make $\underline{k}$ out of $\underline{n}$ indep choices of $0, 1, 2, 3$

$k=0$: $\binom{n}{0}\binom{n}{0} = 1$

$k=1$: $\underset{n}{\binom{n}{0}\binom{n}{1}} + \underset{0}{\binom{n}{1}\binom{n}{-1}}$



$Q(n,k) = Q(n-1,k) + Q(n-1,k-1) + Q(n-1,k-2) + Q(n-1,k-3)$

$\sum_{\#2\text{'s}=0}^{k} \binom{n}{j} \times$ #ways to make $k-2j$ out of remaining $n-j$ guys, each $0, 1, 3$

pick j to be 2's. $0, 1, 2, 3$.

Distribute $+2$ to which ones?

$\sum_{j=\#\geq 2=0}^{k} \binom{n}{j} \times$ #ways to make $k-2j$ out of giving $+1/+0$ to each of $\underline{n}$ buckets $\binom{n}{k-2j}$

1992/B3

$x, \frac{x^2+y^2}{2}, \frac{\left(\frac{x^2+y^2}{2}\right)^2+y^2}{2}, \dots$ Converge to $\underline{c}$. $\frac{c^2+y^2}{2} = c$.

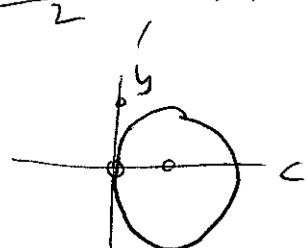
$(1,1) \to \frac{1}{2}, 0), (\frac{1}{8}, 0) \to (9?)$

$(2,1) \to \frac{1}{2} \to \frac{1}{8} + \frac{1}{2} \to \frac{1}{5} + \frac{1}{2} \to \frac{3}{4} \to$

$c^2 - 2c + y^2 = 0$. Needs soln for c, y.

$(c-1)^2 + y^2 - 1 = 0$.

$(c-1)^2 + y^2 = 1$.



Check $(a_n - 1)^2 + y^2$.

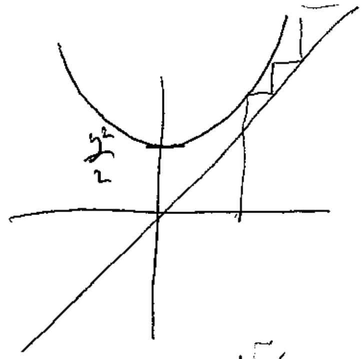
so if x is on circle, it works.

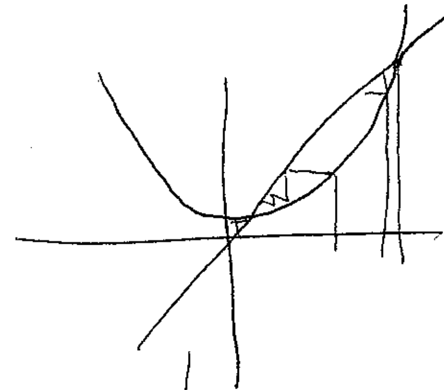
$(a_{n+1} - 1)^2 + y^2$ is... $\left(\frac{a_n^2+y^2}{2} - 1\right)^2 + y^2 = \frac{(a_n^2+y^2)^2}{4} - (a_n^2+y^2) + 1 + y^2$.

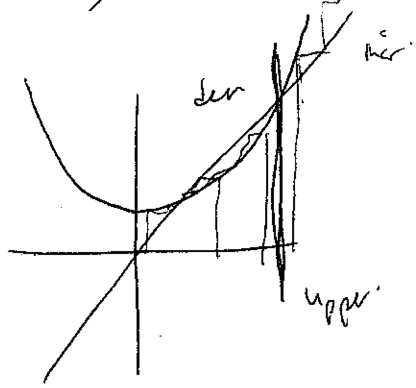
1992/B3

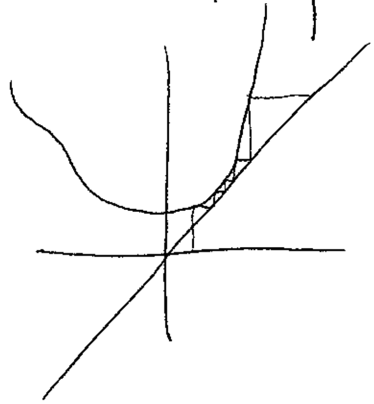
2011-10-03 (B)

$x \to \frac{x^2+y^2}{2}$









when is $\cancel{\frac{x^2+y^2}{2}}$

$\frac{x^2+c^2}{2} = x$ only one soln?

$x^2-2x+c^2=0$

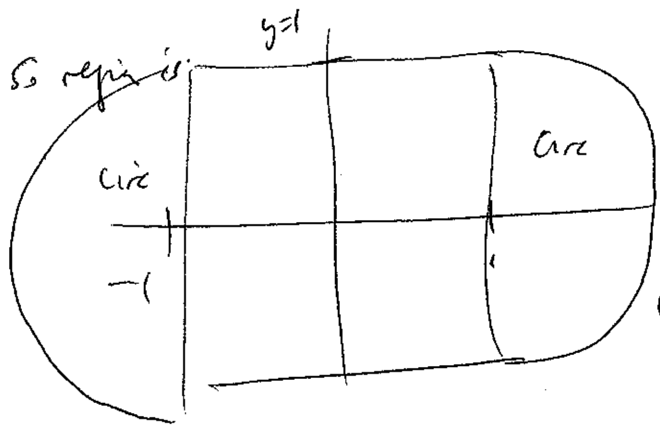
when $c=1$.

Else solns are $x=\frac{2\pm\sqrt{4-4c^2}}{2}$

So take $\frac{2+\sqrt{4-4c^2}}{2}$

$=1+\sqrt{1-c^2}$.

So region is:



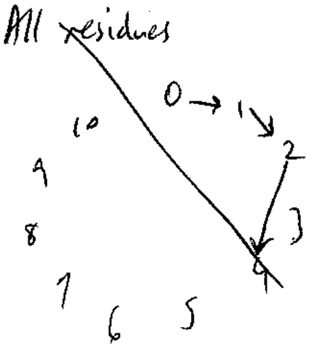
$\pi+4$.

USAMO 1991/3 2011-10-04

(A)

~~$x \mapsto 2^x \bmod m$~~ not how it goes

All ~~residues~~



all vtxs have out-degree 1

who loops?

2

If true for all p:

$2^{p-1} \equiv 1 \pmod p$

$2^p \equiv 2 \pmod p$

$2^{2^{2^x}} \equiv x \pmod m$

possible?

$2^{2^x} \equiv x \pmod m$

2^x if primitive root? if then loop overall

Induction on m: if mod k true

m even: $m = 2^a b$. get same mod b and mod 2^a obviously ends 0.

m odd: $2^{\phi(m)} \equiv 1 \pmod m$ Eventually repeats

And eventually const. for $\phi(m)$ ... get ~~...~~ since cyclic grp

gets $2^{\phi(m)k+r}$ eventually

$m = 3$:

(any ~~...~~)^p ~~...~~ Pick

$\binom{p}{i,j,k} = \binom{p}{i}\binom{p-i}{j}$

$= \frac{p!}{i!(p-i)!}\frac{(p-i)!}{(p-i-j)!j!}$

USAMO 1983/4

odd odd ^ = bold sequence

$r, \quad s, \quad r+s \quad G+s = G\cdot\left(1+\frac{s}{G}\right)$

$G \quad G$

Greatest odd divisor G.

so greatest odd common divisor of r, s always divides in.

Still G divides. Could be nicer?

if not constantly then strictly decreasing over time.

Euler: ~~$\mathbb{Z}$~~$_n^x$ cardinality ~~length~~ Eventually $n \mid 1 - ~~2~~x$.

$(n, 6)$ 1 2 3 4 5 6 repeat ~~divisible~~ elsewise get 6 earlier.

$\mathbb{Z}_6$ mod 6 no inverse

$5 \times 1 = 5$
$2 = 4$
$3 = 3$ Must hit all
$4 =$

Grp: 1. inverse closed. (no 6) (mod) has (no 6)

~~cyclic~~ all elts divide size of grp
orders

$1, a, a^2, a^3, a^4, \ldots$

2011-10-04

$(x-y)^{p-1} = \left(x^{p-1} + a_{p-2}x^{p-2}y + a_{p-3}x^{p-3}y^2 + \cdots + a_1 x y^{p-2} + y^{p-1}\right) * (x-y)$

$(x-y)^p = x^p - y^p$

Coeff of $x^{p-1}y$ is: $a_{p-2} - 1 = 0$.

Coeff of $x^{p-2}y^2$ is $a_{p-3} - a_{p-2} = 0$. OK.

USAMO 1993/4

greatest odd divisor.

$r \quad s$ $\qquad 1 \quad 17 \quad 9. \quad 13 \quad 11 \quad 3 \quad 7 \quad 5 \quad 3 \quad 1. \quad 1. \ldots$

$1 \quad k$

$r \quad G \quad G$ is greatest odd divisor of $r+G$.

so $G \mid r$ too.

So any final G also divides prev.

$x \quad y \quad z$

$\uparrow$ G divides $\quad \uparrow$ G divides

$z \mid x+y$; $G \mid G \Rightarrow G \mid x$ too.

(1) Why it converges? Since $\underset{\text{odd}}{x} \quad \underset{\text{odd}}{y} \quad$ odd divisor of $\frac{x+y}{2}$ $\quad$ odd divisor of $\frac{y+z}{2}$.

$z < \max\{x, y\}$. $\quad \le \max\{y, z\}$.

if $x \neq y$. Then next pair has strictly smaller max.

And note that if $\underset{\text{odd}}{G} \mid x$, $\underset{\text{odd}}{G} \mid y$, then $G \mid x+y$ too. So result limit is mult of G

Also, if we back trace $\quad x \quad y \quad z$

$G \Leftarrow G \quad G$ since $G \mid z \mid x+y$, $G \mid y \Rightarrow G \mid x$.

✓.