7. Inequalities

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1 Classical results

AM-GM. For any non-negative reals $x_1, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n}.$$

Rearrangement. For any reals $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, and any reordering $y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}$,

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Cauchy-Schwarz. For any reals $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \le (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Jensen. For any convex function f, and any reals $x_1, \ldots, x_n$,

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

2 Problems

Warm-up. Over the nonnegative reals, determine the minimum values of (i) $x + \frac{4}{x}$ and (ii) $x^2 + \frac{4}{x^2+3} + 3$.

USAMO 1997/4. A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

IMO 1994/1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$\frac{a_1 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

Putnam 2001/A6. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

From DNA nanotechnology. Define the functions

$$\chi(K) = \frac{(a+1)cK + 1}{2cK} - \frac{\sqrt{(a-1)^2 c^2 K^2 + 2(a+1)cK + 1}}{2cK}$$
$$Q(K) = \frac{\chi(zK)}{\chi(K)}.$$

Prove that for any $a, z \ge 1$ and $c, K > 0$, we always have $Q(K) \le z$.