Special Putnam Training

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Putnam 1997/B1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. **Note that this is not the same as the "fractional" part of x, so this is not standard notation. For example, $\{1.7\} = 0.3$.** For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

Here, $\min(a, b)$ denotes the minimum of a and b.

Putnam 1999/A2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Putnam 2000/A3. The octagon $ABCDEFGH$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $ACEG$ is a square of area 5, and the polygon $BDFH$ is a rectangle of area 4, find the maximum possible area of the octagon.

Putnam 2005/A4. Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Putnam 2005/A5. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx.$$

Putnam 2005/A6. Let $n \geq 4$ be given, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are P_i. What is the probability that at least one of the vertex angles of this polygon is acute?