Special Putnam Training

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Putnam 1995/A1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, not necessarily distinct, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T, and that the product of any three (not necessarily distinct) elements of U is in U, show that at least one of the two sets T, U is closed under multiplication.

Putnam 2005/B2. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Putnam 1995/B3. To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate

$$\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50.$$

Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Putnam 1995/B4. Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \ldots}}}.$$

Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where a, b, c, d are integers.

Putnam 2005/B5. Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \qquad \text{(identically)},$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

Putnam 2005/B6. Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$