2004/A3 2010-11-15

(1)

$$u_{n+3}u_n = u_{n+1}u_{n+2} + n!$$

$u_0 = 1$

$u_1 = 1$

$u_2 = 1$

$u_3 = 1\cdot 1 + 0! = 2$

$u_4 = 1\cdot 2 + 1! = 3$

$u_5 = 2\cdot 3 + 2! = 8$

$u_6 = \frac{1}{2}\cdot\left[3\cdot 8 + 3!\right] = \frac{1}{2}\left[3\cdot 8 + 3\cdot 2\right] = 3\cdot(4+1) = 15.$

$u_7 = \frac{1}{3}\cdot\left[8\cdot 15 + 4!\right] = 8\cdot 5 + 4\cdot 2 = 48$

$u_8 = \frac{1}{8}\cdot\left[15\cdot 48 + 5!\right] = 15\cdot 6 + 5\cdot 3 = 105$

$u_9 = \frac{1}{15}\cdot\left[48\times 105 + 6!\right] = \frac{1}{15}\left[7! + 6!\right] = (7+1)\frac{6!}{15} = (7+1)\frac{6!}{3\times 5} = (8)\times 6\times 4\times 2$

2×4

3×5

4×8

$\boxed{\begin{matrix}2\times 4\times 6\\ 5\times 3\times 7\end{matrix}}$

So... $\boxed{u_n = (n-1)\times(n-3)\times(n-5)\times\cdots \text{ until } 1.}$

Induction:
$$u_{n+3} = \frac{1}{(n-1)(n-3)\cdots}\left[(n+1)(n)\cdots 1 + n!\right]$$
$$= \frac{1}{(n-1)(n-3)\cdots}\left[(n+1)+1\right] n! = (n+2)\,n\,(n-2)\cdots \quad ✓.$$

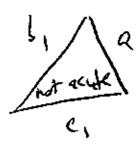
2004/A2

Fix a_2, b_2, c_2. Find ~~maxim~~ maximizer of area for (a_1, b_1, c_1) st. all $a_1 \le a_2$ etc

(exists since area is continuous fn of the side lengths, and on cpct space

Define area $= 0$ if Δ not exists)

Use extremality.

If side $a_1 < a_2$, (there's space), then must have 

else can increase area $\frac{1}{2} b_1 c_1 \sin\theta$.

Also never have obtuse else reducing angle increases area, but shortens a_1 (Law Cosines)

$\Rightarrow$ Any $a_1 < a_2 \Rightarrow 90^\circ$ angle. But so can't have 2 slack.

So say $b_1 = b_2$, $c_1 = c_2$, $a_1 < a_2$ and $\perp$. $\Rightarrow a_2$ gives obtuse ✱.

2000/A1

Clearly ≥ 0, and in fact >0 since sum of positive squares.

Upper bound: $\sum^n x_i^2 \leq \left(\sum^n x_i\right)^2$ if all positive, since extra cross terms.

~~Actually~~

So $\lim \sum^n x_i^2 \leq \lim \left(\sum^n x_i\right)^2 = A^2$.

Can it be equal?

Actually, $\sum^n x_i^2 \leq \left(\sum^n x_i\right)^2 - x_1 x_2$ positive

$\downarrow \lim$

$\sum^\infty x_i^2 \leq A^2 - x_1 x_2$ positive

So $\sum x_i^2 \in (0, A^2)$.

Show how to achieve any of them

Geometric series ~~$\frac{c}{r^i} = x_i$ $\sum x_i = \frac{c}{1} + \frac{c}{r} + \frac{c}{r^2} + \cdots = \frac{c}{1-\frac{1}{r}}$~~

$\sum x_i = c + cr + cr^2 + \cdots = \frac{c}{1-r} = A.$

$c = A(1-r)$

For any $r<1$, we get an initial c

But $\sum x_i^2 = c^2 + c^2r^2 + c^2r^4 + c^2r^6 + \cdots = \frac{c^2}{1-r^2} = A^2 \frac{(1-r)^2}{1-r^2} = A^2 \cdot \frac{1-r}{1+r}$.

~~$= A^2\left[\frac{1-2r+r^2}{1-r^2}\right]$~~

$r \to 1$: get $\to 0$

$r \to 0$: get $\to 1$.