Putnam ADV-C

2010-10-06

(A)

2005/A1 Strong induction

Biggest power of 3.

Right: if N is even, can reduce to ok.

So odd. Take out big power of 3.

$$N = 3^k + \underbrace{M}_{\text{even}}$$

Then take $\frac{M}{2}$, and decompose that, add back 2

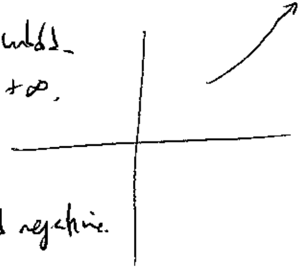
Only need to make sure don't have 3^k dividing one of the $2 \times z$ in $\frac{M}{2}$.

But then $z \geq 3^k \Rightarrow 2z \geq 2 \cdot 3^k \Rightarrow M \geq 4 \cdot 3^k$, wasn't biggest power of 3. ※

1997/B2 $f + f'' = -x\, g\, f'!$ Show $|f|$ bdd.

g always ≥ 0.

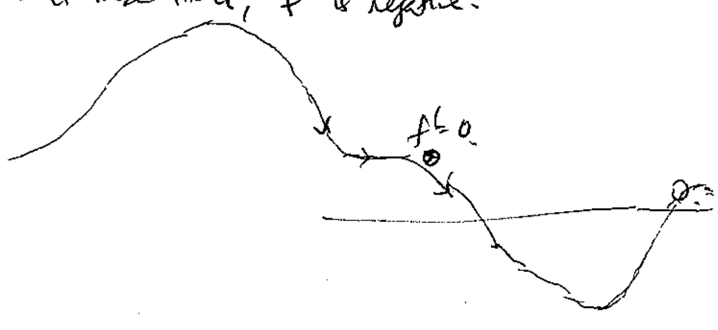
Say f unbdd. as $x \to +\infty$.



$\Rightarrow f''$ unbdd negative.

to $+\infty$. Reaches arbitrary height, and must have times it is then

Then at those times, f'' is negative.



2001/A2 P_n = odd heads after $\underline{n}$.

$$P_1 = \frac{1}{2+1} = \frac{1}{3}.$$

$$P_{n+1} = P_n\left(1 - \frac{1}{2(n+1)+1}\right) + (1-P_n)\left(\frac{1}{2(n+1)+1}\right)$$

$$= P_n - \frac{P_n}{2n+3} + \frac{1}{2n+3} - \frac{P_n}{2n+3}$$

$$= P_n - \frac{2}{2n+3}P_n + \frac{1}{2n+3} = \frac{2n+1}{2n+3}P_n + \frac{1}{2n+3}.$$

n	P_n
1	$\frac{1}{3}$
2	$\frac{3}{5} \times \frac{1}{3} + \frac{1}{5} = \frac{2}{5}$
3	$\frac{5}{7} \times \frac{2}{5} + \frac{1}{7} = \frac{3}{7}$
4	$\frac{7}{9} \times \frac{3}{7} + \frac{1}{9} = \frac{4}{9}$
n	$\frac{n}{2n+1}$

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2003/B3

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \dots, \lfloor \tfrac{n}{i} \rfloor\}$$

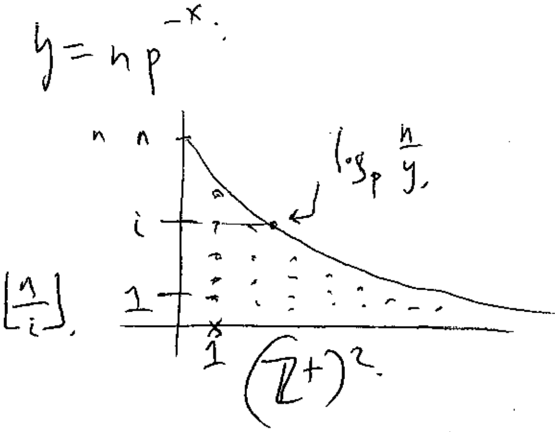
How many p in LHS?

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots + \left\lfloor \frac{n}{p^t} \right\rfloor + \dots$$

How many p in RHS?

For i: it is the max amt of p in up to $\lfloor \frac{n}{i} \rfloor$.

So $\left\lfloor \log_p \lfloor \frac{n}{i} \rfloor \right\rfloor$.



$$\sum_{i=1}^{n} \left\lfloor \log_p \lfloor \tfrac{n}{i} \rfloor \right\rfloor = \left\lfloor \log_p \left(\tfrac{n}{i}\right) \right\rfloor$$

only change when $\frac{n}{i}$ crosses $\mathbb{Z}$ anyway.