Polynomials Putnam 2010-09-21 (1)

P(0) P(1) P(2) — prime

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

prime

Plug in p: then $a_n p^{n-1} + a_{n-1} p^{n-2} + \cdots + a_1 + 1 = 1$ mod

Plug in kp: $a_n k^n p^{n-1} + a_{n-1} k^{n-1} p^{n-2} + \cdots + a_1 k + 1 = 1$ mod

$\forall k \in \mathbb{Z}$ $b_n k^n + \cdots + b_1 k + 1 = 1$ ~~mod~~ everywhere

So ~~constant~~ $\Rightarrow$ $\not\equiv$.

a	$\lfloor a \rfloor$	$\lfloor 2a \rfloor$
$[0, \frac{1}{2})$	0	0
$[\frac{1}{2}, 1)$	0	1
$[1, 1.5)$	1	2
$[1.5, 2)$	1	3
$[2, 2.5)$	2	4
	2	5
	3	6
	3	7
	⋮	
$[-0.5, 0)$	-1	-1
$[-1, -0.5)$	-1	-2

$\cancel{(2x - y)(2x + 1 - y)}$

$(y - 2x)(y - (2x+1))$

Even function

$P(x) = a_n x^n + \cdots$

Suppose odd power

Then can be unbounded in both directions

If leading coeff is (+) even, then

Even function: $P(z) = P(-z)$

Subtract: $\forall z: \sum 2 a_{odd} z^{odd} = 0$.

~~for large~~

Why all coeffs 0? Since leading term would dominate and grow

Specifically so all odd coeffs 0.

$P(z) = Q(z^2) = a(z^2 - r_1)(z^2 - r_2) \cdots (z^2 - r_k)$

$= a(z + \sqrt{r_1})(z - \sqrt{r_1}) \cdots (z + \sqrt{r_k})(z - \sqrt{r_k})$

$\cancel{Q(z) = \sqrt{a}\,(z + \sqrt{r_1})(z + \sqrt{r_2}) \cdots (z + \sqrt{r_k})}$

$= a(\sqrt{r_1} - z)(\sqrt{r_1} + z) \cdots (\sqrt{r_k} + z)(\sqrt{r_k} - z)$

$Q(z) = \sqrt{a}\,(\sqrt{r_1} - z) \cdots (\sqrt{r_k} - z)$ ✓.

$Q(-z) = \sqrt{a}\,(\sqrt{r_1} + z) \cdots (\sqrt{r_k} + z)$

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②

Odd degree $\mathbb{R}$ coeffs

$\mathbb{R}$ Roots for $P(x)$ are $r_1, r_2, \ldots, r_m$. Need m roots for $P(P(x))$
Each has a value that goes to it
WLOG

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

~~Suppose $x \geq$~~

Still $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ hits all values

~~Easy $x \geq$~~

When is $|x^n| \geq \frac{1}{2} |a_{n-1} x^{n-1} + \cdots + a_0|$

$$|x| \geq \frac{1}{2} \left| a_{n-1} + \frac{a_{n-2}}{x} + \frac{a_{n-3}}{x^2} + \cdots + \frac{a_0}{x^{n-1}} \right|$$

So once $x \geq 1 + \frac{1}{2}\left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|\right)$ Constant.

$$P(x) \geq \frac{1}{2} x^n \nearrow +\infty$$

Also if $x \leq -$ constant $P(x) \leq -\frac{1}{2} x^n \searrow -\infty$

$c_n r^n + \cdots + c_0$ is $\mathbb{Z}$. $(P(r) - c_0)$

$$c_n r^{n-1} + \cdots + c_3 r^2 + c_2 r = \frac{P(r) - c_1 r - c_0}{r} = \frac{P(r)}{r} - c_1 - \frac{c_0}{r} = \frac{P(r) - c_0}{r} - c_1 \in \mathbb{Z}$$

ok.

Next: $\frac{P(r) - c_2 r^2 - c_1 r - c_0}{r^2} = c_2 + \frac{P(r) - c_1 r - c_0}{r^2}$ $\frac{p^2}{q^2}$ Need $p^2 \mid c_1 p q + c_0 q^2$

Need $r = \frac{p}{q}$.

$$c_n \frac{p^n}{q^n} + \cdots + c_0 = 0 \rightarrow c_n p^n + c_{n-1} p^{n-1} q + \cdots + \underbrace{c_1 p q^{n-1} + c_0 q^n}_{p^2 \mid} = 0$$

$\rightarrow p \mid c_0$

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(3)

$f(1) \mid f(f(1)+1)$

$a_n x^n + \dots + a_0$

$a_n + \dots + a_0 \mid f(a_n + \dots + a_0 + 1)$

$a_n [f(k)+1]^n + a_{n-1}[f(k)+1]^{n-1} + \dots + a_0$

Remainder modulo $f(k)$ is 1

$\Leftrightarrow a_n + a_{n-1} + \dots + a_0$

$f(k) \mid f(1)$

$k=1$ satisfies this.

And always $f(k) > f(1)$ since $\mathbb{Z}^+$-coeffs

$-\frac{b}{a} = r_1 + r_2 + r_3 + r_4$

$\frac{c}{a} = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4$

$= r_1 r_2 + r_3(r_1+r_2) + r_4(r_1+r_2) + r_3 r_4$

$\mathbb{Q}$ $\mathbb{Q}$

$(r_1+r_2)(r_3+r_4)$

So $r_1 r_2 + r_3 r_4 \in \mathbb{Q}$

$r_1 + r_2$ $\in \mathbb{Q}$

$(r_1+r_2)^2 + (r_3+r_4)^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2r_1r_2 + 2r_3r_4$

$r_1^2 + r_2^2 + 2r_1r_2 \in \mathbb{Q}$

$(r_1 + r_2 - r_3 - r_4)^2 = \sum r_i^2 + 2[r_1r_2 + r_3r_4 + \text{cross terms}]$

$\hookrightarrow \sum r_i^2$ are $\mathbb{Q}$

$(r_1 + r_2 - r_3 - r_4)$

$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4$

$\mathbb{Q} \ni r_1 r_2 (r_3 + r_4) + r_3 r_4 (r_1 + r_2)$

a B b A

$aB + bA$

$a+b$

A, B all $\mathbb{Q}$

~~$aA + bA \in \mathbb{Q}$~~

~~$aB + bB \in \mathbb{Q}$~~

$aA + bA \in \mathbb{Q}$

$a(B-A) \in \mathbb{Q}$ ✓