The Hypergraph 2-colouring Threshold

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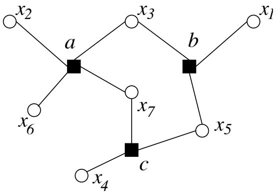
Random CSPs

Constraint Satisfaction Problems ("CSPs")

- $x_1, \ldots, x_n$: *variables* with a finite domain D ("spins").
- $C_1, \ldots, C_m$: *constraints* binding a small number of variables each.
- Goal: an assignment

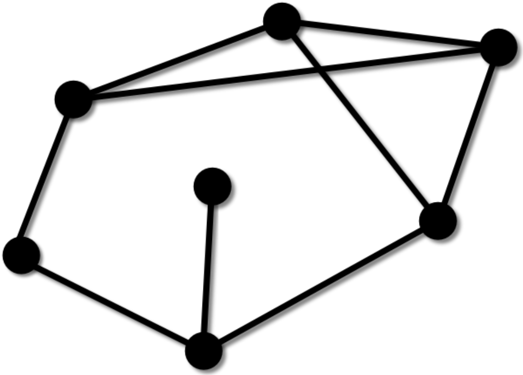
$$\sigma : \{x_1, \ldots, x_n\} \to D$$

that satisfies *all* constraints.



Example: graph colouring

- $G =$ a graph with n *vertices* and m *edges*.
- Question: does G admit a *3-colouring*?



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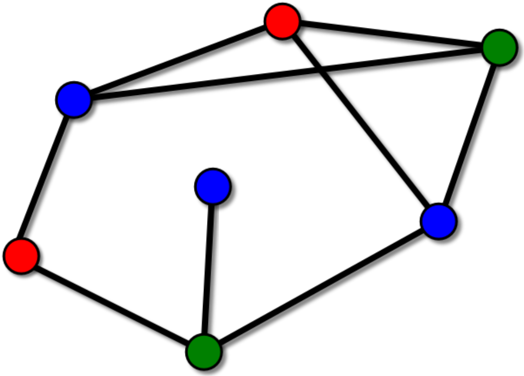

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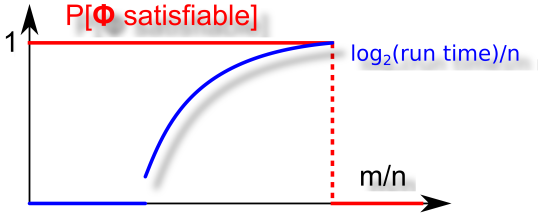
Kirkpatrick, Selman (experimental) [Science 1994]

There occurs a sharp satisfiability phase transition.



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- Existence of *non-uniform* thresholds *[Friedgut 1999]*
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- Pinning down the *thresholds* (random k-SAT, graph colouring, ...)
- (*Computational* aspect.)

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- (*Computational* aspect.)

- Cavity method: "Survey propagation" *[Mézard, Parisi, Zecchina 2002]*
- The condensation transition *[KMRSZ 2007]*
- *Universal* picture (random k-SAT, graph colouring, ...)

The statistical mechanics perspective

- Phase transitions in glasses hypothesized by *Kauzmann* (1948).
- Mean-field models of disorered systems (such as glasses).
- This work: first *proof* of condensation in a "diluted mean-field model".

Random CSPs

This work: random hypergraph 2-colouring

- Pinning down the *threshold* in a problem with condensation.
- Rigorous approach to condensation.

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Known thresholds

- Random 2-SAT *[Chvátal, Reed'92; Goerdt'92]*
- Random 1-in-k-SAT *[Achlioptas, Chtcherba, Istrate, Moore'01]*
- Random k-XORSAT *[Dubois, Mandler'02]*
- Uniquely extendible problems *[Connamacher, Molloy'04]*
- Random k-SAT with $k > \log_2 n$ *[Frieze, Wormald'05]*

Random Hypergraph 2-colouring

Random Hypergraphs

- $V = \{v_1, \ldots, v_n\}$: vertices.
- $\mathcal{H} =$ random *k-uniform hypergraph* with m edges.
- Let $r = m/n$ be *fixed* while $n \to \infty$.

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Hypergraph 2-colouring

- Is there $\sigma : V \to \{\bullet, \bullet\}$ s.t. no edge is *monochromatic*.
- NP-hard in the *worst case*.

The partition function

- Let $\beta > 0$ be a parameter *("inverse temperature")*.
- For $\sigma : V \to \{\bullet, \bullet\}$ let

$$w(\sigma) = \#\text{monochromatic edges in } \mathcal{H} \text{ under } \sigma.$$

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$$Z_\beta = \sum_{\sigma : V \to \{\color{red}{\bullet}, \color{blue}{\bullet}\}} \exp(-\beta \cdot w(\sigma)).$$

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- Goal: to find

$$(\beta, r) \mapsto \lim_{n \to \infty} \frac{1}{n} \mathrm{E}\left[\ln Z_\beta\right] \geq 0.$$

- *Bayati, Gamarnik, Tetali 2010:* the limit exists for any $0 < \beta < \infty$.

The partition function

Zero temperature

- *Special case:* $\beta = \infty$.
- Set
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Conjecture

The limit $\lim_{n\to\infty} \frac{1}{n} \mathrm{E}\left[\ln\left(1 + Z_\infty\right)\right]$ exists for any $r > 0$.

This implies the *"sharp threshold conjecture"*.

The partition function

Phase transitions

- a point (β, r) where the limit is non-analytic.
- a density r where the *zero temperature* limit is non-analytic.

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Key questions

- Do *one or more* phase transitions exist?
- *Zero temperature*: the 2-colouring threshold r_{col}, plus ...?

Results

Theorem [ACO, Zdeborová 2012]

1. The *zero temperature* limit is non-analytic at

$$r_{cond} = 2^{k-1}\ln 2 - \ln 2 + o_k(1) \quad \textit{and} \quad r_{col} > r_{cond}.$$

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- *Zero temperature:* (at least) two phase transitions.
- *Low temperature:* at least one.

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Theorem [ACO, Panagiotou 2012]

We have $r_{col} = 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{4} + o_k(1)$.

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Density	What's happening?
$2^{k-1}\ln 2 - \frac{\ln 2}{2} - \frac{1}{2} + o_k(1)$	"vanilla" second moment *[AM'02]*
$2^{k-1}\ln 2 - \ln 2 + o_k(1)$	*phase transition* ("condensation")
$2^{k-1}\ln 2 - \frac{\ln 2}{2} - \frac{1}{4} + o_k(1)$	2-colouring *threshold*
$2^{k-1}\ln 2 - \frac{\ln 2}{2} + o_k(1)$	*first moment* upper bound

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- By convexity, we have

$$\frac{1}{n} \mathrm{E}[\ln Z] \leq \frac{1}{n} \ln \mathrm{E}[Z].$$

- Hence,

$$r_{col} \leq 2^{k-1} \ln 2 - \frac{\ln 2}{2} + o_k(1).$$

The second moment

Vanilla second moment [Achlioptas, Moore 2002]

- For $r \leq 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{2} + o_k(1)$ we have

$$\mathrm{E}\left[Z^2\right] \leq C \cdot \mathrm{E}\left[Z\right]^2.$$

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- Let $Z(d) = \#\text{colourings } \tau \text{ with } \mathrm{dist}(\sigma, \tau) = d$.
- Then

$$\mathrm{E}\left[Z|\sigma\right] = \sum_{d=0}^{n} \mathrm{E}\left[Z(d)|\sigma\right].$$

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- Further,

$$\begin{aligned}
\frac{1}{n}\ln \mathrm{E}\left[Z(\alpha n)|\sigma\right] &= H(\alpha) + E(\alpha), \text{ with} \\
H(\alpha) &= -\alpha\ln(\alpha) - (1-\alpha)\ln(1-\alpha), \\
E(\alpha) &= r \cdot \ln\left[1 - \frac{1-\alpha^k-(1-\alpha)^k}{2^{k-1}-1}\right].
\end{aligned}$$

The second moment

- We have $\ln \mathrm{E}[Z|\sigma] \sim \max_{0 \le \alpha \le 1} \ln \mathrm{E}[Z(\alpha n)|\sigma]$.
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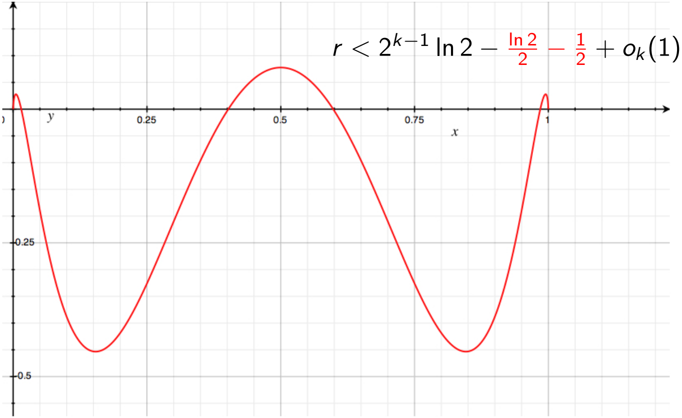

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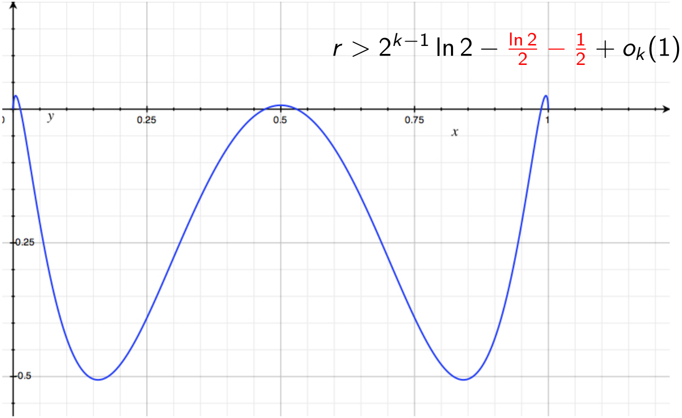

The solution space

Stat mech hypothesis [Krzkala et al.: PNAS 2007]

- Let $\mathcal{S}(\mathcal{H}) = \{\text{all 2-colourings of } \mathcal{H}\}$.
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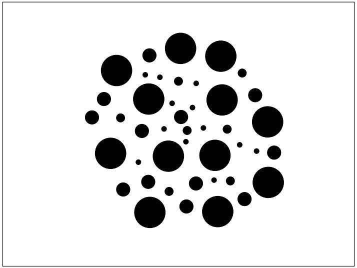

- For $r < 2^{k-1}\ln 2 - \ln 2 + o_k(1)$, the set shatters into tiny *clusters*.
- Each cluster size is *exponentially small*.

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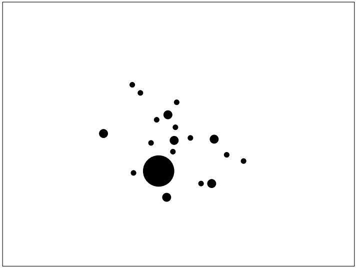


- For $r > 2^{k-1}\ln 2 - \ln 2 + o_k(1)$, the set condenses.
- A *bounded* number of clusters dominate.

The solution space

The "shape" of the clusters

- Clusters are characterised by frozen vertices.
- Frozen vertices govern the *cluster size:*

$$\frac{1}{n}\log_2\{\text{cluster size}\} \sim 1 - \frac{\#\text{frozen}}{n}.$$

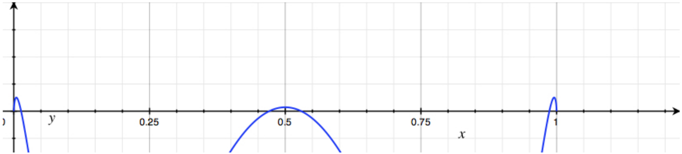
Rigorous work

- Achlioptas, Ricci-Tersenghi 2006.
- Achlioptas, ACO 2008.
- Molloy 2012

Second moment redux

Question

Why does the second moment *break* before condensation?



- In the plot of $\alpha \mapsto \frac{1}{n} \ln \mathrm{E}\left[Z(\alpha n) | \sigma\right] \ldots$
- ...think of the *max near* 0 as the expected cluster size.
- Driven up by fluctuations in the number of frozen vertices.

Second moment redux

Controlling the cluster size [ACO, Zdeborová 2012]

- The second moment breaks because...
- ... *exceptional* formulas drive up the expected cluster size.

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$$Z_{good} = \#\text{colourings whose cluster size is} \leq \mathrm{E}[Z].$$

- Then $\mathrm{E}[Z_{good}] \sim \mathrm{E}[Z]$ if $r \leq 2^{k-1}\ln 2 - \ln 2$.

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- We have

$$\mathrm{E}\left[Z_{good}^2\right] \leq C \cdot \mathrm{E}[Z_{good}]^2$$

for any

$$r \leq 2^{k-1}\ln 2 - \ln 2 = \text{predicted } \textit{condensation} \text{ point}.$$

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Corollary [ACO, Zdeborová 2012]

- For $r \leq 2^{k-1}\ln 2 - \ln 2 + o_k(1)$ we have

$$\mathrm{E}\left[\ln Z\right] \sim \ln \mathrm{E}\left[Z\right] = \ln 2 + r \cdot \ln(1 - 2^{1-k}).$$

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- There is shattering. . .
- . . . and the *cluster size* of a random 2-colouring is w.h.p.

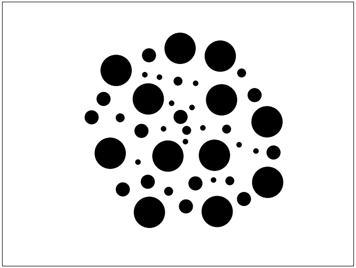
$$\frac{1}{n}\log_2\{\text{cluster size}\} \sim \exp\left(-\frac{kr}{2^{k-1}-1}\right).$$

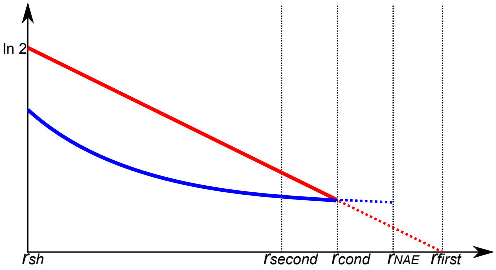
The entropy crisis

A phase transition [ACO, Zdeborová 2012]

Let's plot the functions

$$r \mapsto \frac{1}{n}\mathrm{E}\left[\ln Z\right] \qquad \text{and} \qquad r \mapsto \frac{1}{n}\mathrm{E}\left[\ln \{\text{cluster size}\}\right].$$

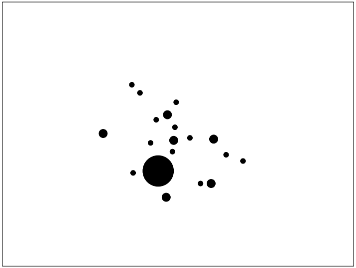


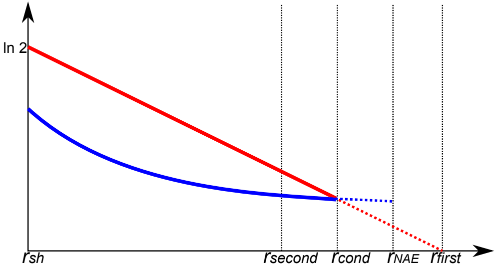


Into the condensation phase

Targeting small clusters [ACO, Panagiotou 2012]

- Idea: count solutions in *small clusters only*.
- These (supposedly) remain abundant and *well-separated*.





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- *Key parameter:* #frozen vertices.

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- *Key parameter:* #frozen vertices.
- *Proxy:* v is blocked in σ if flipping v leaves an edge monochromatic.
- Let $Z_\gamma = \#$colourings with γn blocked vertices.
- The *second moment analysis* for Z_γ succeeds so long as

$$\frac{1}{n}\ln\{\text{cluster size}\} \sim (1-\gamma)\ln 2 \le \frac{1}{n}\ln \mathrm{E}\left[Z_\gamma\right].$$

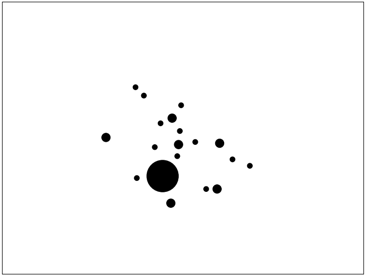
- Optimising over γ gives the threshold.

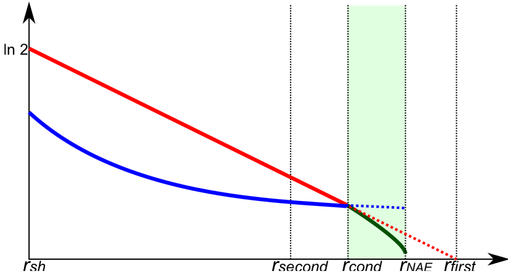
Into the condensation phase

Corollary [ACO, Panagiotou 2012]

Approximate expressions for...

- ...the *partition function* $\frac{1}{n}\mathrm{E}[\ln Z]$,
- ...the *number of clusters* ("complexity").





Conclusion

- Main contricutions:
  - first *improvement* over the "vanilla" 2nd moment from [AM02],
  - first rigorous proof of a *condensation transition* in this kind of model,
  - pinned down the *2-colouring threshold* up to $o_k(1)$.
- Techniques:
  - physics-inspired second moment argument,
  - exploiting the solution space geometry,
  - differential equation, cores, ...
- Open problems:
  - *exact* threshold for any k?
  - extension to graph coloring?